Applied Quantum Mechanics

Chapter 1 problems

LAST NAME FIRST NAME

Problem 1.1
A metal ball is buried in an ice cube that is in a bucket of water.

(a) If the ice cube with the metal ball is initially under water, what happens to the water level when the ice melts?

(b) If the ice cube with the metal ball is initially floating in the water, what happens to the water level when the ice melts?

(c) Explain how the Earth's average sea level could have increased by at least 100 m compared to about 20 000 years ago.

(d) Estimate the thickness and weight per unit area of the ice that melted in (c). You may wish to use the fact that the density of ice is 920 kg m^{-3}, today the land surface area of the Earth is about 148 300 000 km^2 and water area is about 361 800 000 km^2.

Problem 1.2
Sketch and find the volume of the largest and smallest convex plug manufactured from a sphere of radius $r = 1$ cm to fit exactly into a circular hole of radius $r = 1$ cm, an isosceles triangle with base 2 cm and a height $h = 1$ cm, and a half circle radius $r = 1$ cm and base 2 cm.

Problem 1.3
An initially stationary particle mass m_1 is on a frictionless table surface and another particle mass m_2 is positioned vertically below the edge of the table. The distance from the particle mass m_1 to the edge of the table is l. The two particles are connected by a taught, light, inextensible string of length $L > l$.

(a) How much time elapses before the particle mass m_1 is launched off the edge of the table?

(b) What is the subsequent motion of the particles?

(c) How is your answer for (a) and (b) modified if the string has spring constant κ_0?

Problem 1.4
The velocity of waves in shallow water may be approximated as $\upsilon = \sqrt{gh}$ where g is the acceleration due to gravity and h is the depth of the water. Sketch the lowest frequency standing water wave in a 5 m long garden pond that is 0.9 m deep and estimate its frequency.

Problem 1.5
What is the dispersion relation of a wave whose group velocity is (a) half the phase velocity, (b) twice the phase velocity, (c) four times the phase velocity, and (d) the negative of phase velocity?

Problem 1.6
(a) If classical electromagnetism were described in terms of a single *complex* field $\mathbf{G}$ show that Maxwell's equations in free space and in the absence of free charges may written as the complex equations

$$\nabla \cdot \mathbf{G} = 0$$

and

$$i\frac{\partial \mathbf{G}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times \mathbf{G}$$

where $\mathbf{G} = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{D}}{\sqrt{\varepsilon_0}} + i\frac{\mathbf{B}}{\sqrt{\mu_0}}\right)$

(b) Show that the energy flux density in the electromagnetic field given by the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{-i}{\sqrt{\varepsilon_0 \mu_0}}(\mathbf{G}^* \times \mathbf{G})$$

(c) If the field **G** is purely real, what is the value of **S**?

(d) Show that the electromagnetic energy density is $U = |\mathbf{G}|^2$.

(e) How would Maxwell's equations be modified if magnetic charge g (magnetic monopoles) were discovered? Derive an expression for conservation of magnetic current and write down a generalized Lorentz force law that includes magnetic charge. Write Maxwell's equations with magnetic charge in terms of a field $\mathbf{G}$.

Problem 1.7

(a) The capacitance of a small metal sphere in air is $C_0 = 1.1 \times 10^{-18}$ F. A thin dielectric film with relative permittivity $\varepsilon_{r_1} = 10$ uniformly coats the sphere and the capacitance increases to 2.2×10^{-18} F. What is the thickness of the dielectric film and what is the single electron charging energy of the dielectric coated metal sphere?

(b) The dielectric coated metal sphere of part (a) is now coated with metal. What is the new value of the single-electron charging energy for the central metal sphere?

(c) Compare the result in (b) to the charging energy of a metal sphere radius $0.5\text{ nm} < r_0 \leq 10\text{ nm}$ embedded in a dielectric of relative permittivity $\varepsilon_r = 10$ and surrounded by metal shell of internal radius $r_1 = 2r_0$. Plot single-electron charging energy ΔE as a function of r_0.

Problem 1.8

(a) A diatomic molecule has atoms with mass m_1 and m_2. An isotopic form of the molecule has atoms with mass m'_1 and m'_2. Find the ratio of vibration oscillation frequency ω / ω' of the two molecules.

(b) What is the ratio of vibrational frequencies for carbon monoxide isotope 12 ($^{12}C^{16}O$) and carbon monoxide isotope 13 ($^{13}C^{16}O$)?

Problem 1.9

A house in California consumes 10 MW hrs of electricity over a one year period.

(a) What is the minimum vertical distance one would have to lift a 200 000 ton block of concrete to store this amount of energy?

(b) A 25% efficient water driven electric generator is placed at the bottom of a 10 m high waterfall. What is the minimum water flow rate (measured in tons per minute) required to supply the average power consumption of the house?

(c) Iron weights six inches in diameter are attached to a cable and hung in a well of diameter greater than six inches and depth h. The iron weighs 140 kg m^{-1} and the weight of the cable is insignificant. What is the maximum amount of energy that can be stored using this method if $h = 100\text{ m}$ and how much more energy can be stored if $h = 1000\text{ m}$?

Problem 1.10

A one centimeter long linear chain of spherical atoms has nearest neighbor spacing of 0.25 nm.

(a) What is the minimum diameter of a one atom thick disk made of these atoms?

(b) What is the minimum diameter of a sphere made of these atoms?

Problem 1.11

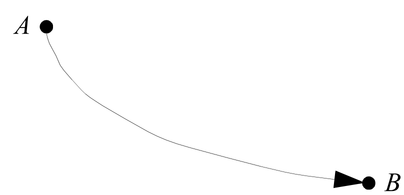
An electromagnetic wave has electric field $\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega)e^{i(k'(\omega)+ik''(\omega))\mathbf{k}\cdot\mathbf{r}}$ where $k'(\omega)$ and $k''(\omega)$ are the real and imaginary parts, respectively, of the frequency-dependent wave number. The wave propagates in a homogeneous dielectric characterized by $\mu_\mathrm{r} = 1$ and complex permittivity function $\varepsilon(\omega) = \varepsilon_0\varepsilon_\mathrm{r}(\omega) = \varepsilon_0(\varepsilon_\mathrm{r}'(\omega) + i\varepsilon_\mathrm{r}''(\omega))$, where $\varepsilon_\mathrm{r}'(\omega)$ and $\varepsilon_\mathrm{r}''(\omega)$ are the real and imaginary parts, respectively, of the frequency-dependent relative permittivity function.

(a) Derive the expression for refractive index $n_\mathrm{r}(\omega) = \sqrt{\frac{1}{2}(\varepsilon_\mathrm{r}'(\omega) + \sqrt{\varepsilon_\mathrm{r}'^2(\omega) + \varepsilon_\mathrm{r}''^2(\omega)})}$.

(b) Introduce absorption coefficient $\alpha(\omega) = 2k''(\omega)$ and show that $\alpha(\omega) = \frac{\omega}{c}\frac{\varepsilon_\mathrm{r}''(\omega)}{n_\mathrm{r}(\omega)}$.

Problem 1.12

A particle moves between two points A and B in a vertical plane as illustrated in the figure below. If acceleration due to gravity is g and velocity is initially zero, find the shape of the frictionless surface on which the particle must move to give a trajectory that takes the shortest time.



Problem 1.13

Materials with negative relative permeability and negative relative permittivity can display negative refractive index. In this situation group velocity is the negative of phase velocity. Suppose a point source of electromagnetic radiation in air is placed at a distance z_1 normal to the surface of a slab of negative refractive index material of thickness $z_2 > z_1$. The value of the negative refractive index material is $n_\mathrm{r} = -1$. Use ray tracing to find the positions at which electromagnetic radiation from the point source is focused to a point. Comment on the statement that this slab of negative index material makes a "perfect lens".

Problem 1.14

An electromagnetic field of wavelength $\lambda_0 = 1500$ nm in free space propagates around the inside circumference of a silica dielectric disk resonator of density $\rho = 2.2\ \mathrm{g\ cm^{-3}}$ and refractive index $n_\mathrm{r} = 1.5$. The disk has radius $R = \frac{0.2}{2\pi}$ mm and thickness $d = 1\ \mu\mathrm{m}$. Electromagnetic field loss in the disk is dominated by surface roughness with average value $\alpha = 0.016\ \mathrm{cm^{-1}}$.

(a) Calculate the lowest natural radial mechanical oscillation frequency of the disk in the absence of light. Young's modulus for the dielectric material is $E_Y = 73$ GPa and the Poisson ratio is $\nu = 0.17$.

(b) Calculate the maximum resonant enhancement in electric field and electric field intensity and the full-width half-maximum (FWHM) in the field intensity frequency spectrum.

(c) Repeat the calculation in (b) but for $R = \frac{0.02}{2\pi}$ mm.

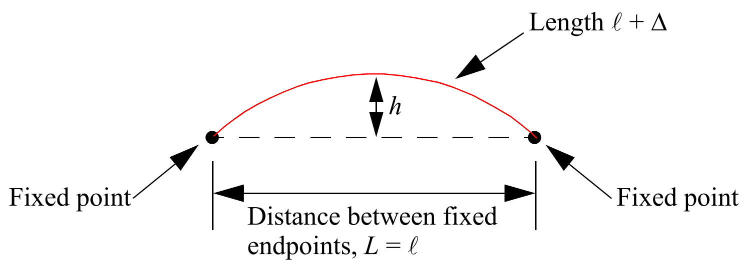
(d) Compare and explain the results obtained in (b) and (c).

(e) $10\ \mu\mathrm{W}$ of optical power at $\lambda_0 = 1500$ nm wavelength is coupled into the dielectric disk in (b). Estimate the force exerted on the disk due to radiation pressure and estimate the change in disk radius. Compare the resulting shift in resonant frequency to the optical FWHM. What optical modulation depth might be achievable in the system?

(f) What is the FWHM of the frequency intensity spectrum resonance if the electromagnetic field propagating around the inside circumference of the dielectric disk resonator experiences a negative refractive index $n_r' = -1.5$ for exactly one half of each round trip? How does this change the results you obtained in part (d)?

Problem 1.15

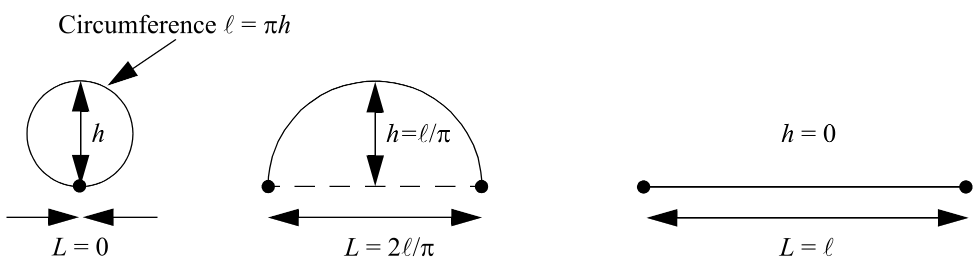
The initial length $\ell = 2\ \mu\text{m}$ of a thin horizontal micro-beam increases by Δ causing the beam to describe the arc of a circle between fixed endpoints separated by distance $L = \ell$.



(a) Calculate and plot the height of the midpoint h as a function of Δ for 0 nm < Δ < 10 nm.

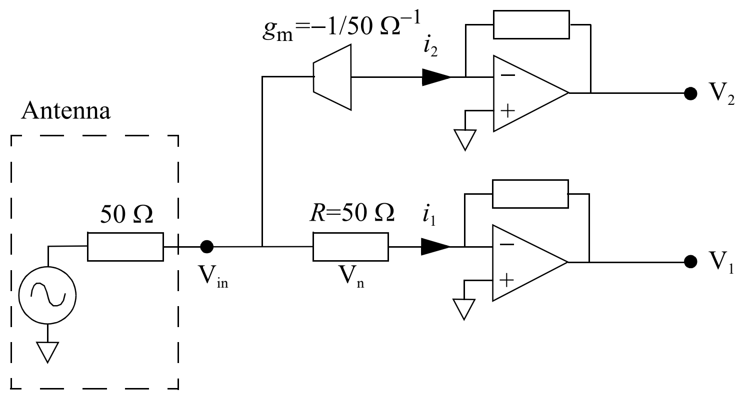
(b) Calculate and plot the mechanical gain $g \equiv h/\Delta$ for 0 nm < Δ < 10 nm.

(c) Setting $\Delta = 0$, find and plot h as a function of fixed endpoint separation L in the continuous range 0 to ℓ. The following shows examples for $L = 0$, $L = 2\ell/\pi$, and $L = \ell$. Find the value of L that gives the maximum value of h.



Problem 1.16

Johnson (*Phys. Rev.* **32**, 97 (1928)) and Nyquist (*Phys. Rev.* **32**, 110 (1928)) showed that thermal fluctuations (whose cause is fundamentally due to interactions between quantized particle states) create RMS voltage noise $V_{\text{RMS}} = \sqrt{4Rk_BT\Delta f}$ in a macroscopic resistor of value R (ohms) at absolute temperature T (kelvin) measured over a frequency bandwidth Δf, so long as the frequencies considered $f \ll k_BT/(2\pi\hbar)$. This noise can limit sensitivity of a RF receiver. Bruccoleri et al. (*IEEE J. Solid-State Circuits*, **39**, 275 (2004)) showed how the following circuit, in which the current-source transconductance amplifier (g_m cell) is an inverter, could be used to cancel thermal noise generated by the input load resistor R.



Explain how this noise cancellation works by evaluating the current i_1 and i_2 for a voltage signal V_{in} at the input and voltage noise V_n generated in the resistor R. What physical principals and conservation laws do you exploit to analyze the circuit? What limits the performance of the noise cancellation circuit?

Problem 1.17

Suppose dipole radiation energy-loss rate $\frac{dU}{dt}$ from an electron of charge e, mass m_0, and acceleration a, obeys the Larmor formula

$$\frac{dU}{dt} = \frac{-2e^2a^2\varepsilon_0\mu_0}{4\pi\varepsilon_0 3c}$$

where c is the speed of light and ε_0 and μ_0 is the permittivity and permiability of free-space respectively.

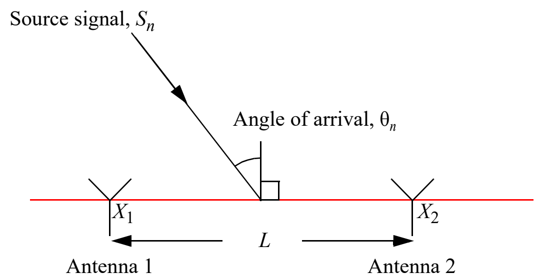
(a) *If* it is possible to describe electron motion around a proton classically and *if* the electron is initially in a circular orbit of radius r around the proton, what is the acceleration and velocity of the electron and how long does it take the electron to complete one round-trip assuming $r = a_B = 0.0529$ nm ?

(b) What is the total (non-relativistic) energy, U, of the electron in (a)?

(c) If the energy loss due to electromagnetic radiation occurs slowly compared to the round-trip time, τ_r, an adiabatic approximation assumes the orbit radius, r, remains almost circular at all times. Use the time-derivative of total electron energy at radius r calculated in (b) and the Larmor formula to find the time it takes the electron to radiate away all its kinetic energy and arrive at the origin, $r = 0$ (corresponding to a classical collapse of the hydrogen atom).

(d) Within the approximation of (c), at what radius is the electron velocity 10% of the speed of light? How do relativistic effects influence your calculation in (c)?

Problem 1.18



Two identical antennas, labeled 1 and 2, are separated in free-space by distance L. If the antennas receive an electromagnetic signal from source S_n that has angular frequency ω_n then, as a function of time t, a unit-amplitude source signal is $S_n(t) = e^{i\omega_n t}$. If the n-th signal $S_n(t)$ is a plane wave and has an angle of arrival θ_n measured anticlockwise from normal incidence then there is a relative phase difference of ϕ_n between the contribution of $S_n(t)$ arriving at antenna 1 and 2. In general the relationship between angle of arrival θ_n of the n-th signal and the phase difference ϕ_n is

$$\phi_n = \frac{2\pi L}{\lambda_n}\sin(\theta_n)$$

for a signal of wavelength $\lambda_n = 2\pi c/\omega_n$,where c is the speed of light. If there are only two plane-wave sources, $S_1(t)$ and $S_2(t)$, then each antenna receives the sum of the two signals and at any given time this sum may be written in matrix form as

$$\mathbf{X} = \mathbf{AS}$$

where

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is the vector describing signal $X_1(t)$ received at antenna 1 and signal $X_2(t)$ received at antenna 2,

$$\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

is the vector describing the sources $S_1(t)$ and $S_2(t)$, and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is the time-independent complex mixing matrix.

(a) Find the matrix elements of the mixing matrix, $\mathbf{A}$.

(b) Find the inverse mixing matrix $\mathbf{A}^{-1}$ and find the conditions when it *not* possible to separate the source signals using $\mathbf{S} = \mathbf{A}^{-1}\mathbf{X}$.

(c) In a typical wireless receiver system implementation the complex signals are separated into their real (in-phase, I) and imaginary (quadrature, Q) components at each antenna relative to a reference local oscillator. This doubles the size of the mixing matrix. Find the matrix elements for the mixing matrix in this case.

(d) Discuss how the ability to separate source signals is changed if the position of the antennas and sources vary in time so that $L = L(t)$ and $\theta_n = \theta_n(t)$?

(e) Can a RF receiver be used to measure electromagnetic field?

Problem 1.19

The Drude model of electrical conduction predicts a zero-frequency (DC) normal metal conductivity $\sigma_0 = e^2 n\tau/m$, where e is the electron charge, n the electron density, m the electron mass, and the characteristic electron collision time is τ. The frequency dependent (AC) conductivity is

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} = \frac{\omega_p^2\varepsilon_0\tau}{1 - i\omega\tau}$$

where $\omega_p = 2\pi f_p$ is the electron plasma frequency and $1/\tau = \omega_\tau = 2\pi f_\tau$ is the electron collision rate. A linearly polarized electromagnetic plane-wave propagating in the x-direction and normally incident on a planar metal interface at $x = x_0$ has electric field $E_x(x) = E_{x0}e^{ikx}$ in the metal, where

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r(\omega)} = \frac{\omega}{c}\sqrt{1 - \frac{\sigma_0\tau}{\varepsilon_0(1+\omega^2\tau^2)} + i\frac{\sigma_0}{\varepsilon_0\omega(1+\omega^2\tau^2)}}$$

and $\varepsilon_r(\omega)$ is the permittivity of the metal. For copper $\sigma_0(\mathrm{Cu}) = 5.9\times10^7\ \mathrm{S\ m^{-1}}$, $n(\mathrm{Cu}) = 8.46\times10^{28}\ \mathrm{m^{-3}}$, $\tau(\mathrm{Cu}) = 25\ \mathrm{fs}$, $f_p(\mathrm{Cu}) = 2600\ \mathrm{THz}$, and $f_\tau(\mathrm{Cu}) = 6.4\ \mathrm{THz}$.

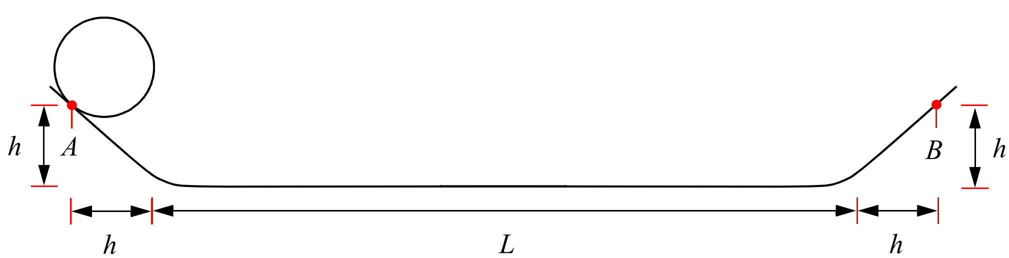
(a) Find an expression for the frequency dependent skin depth $\delta_x(\omega) = 1/(\mathrm{Im}(k(\omega))$ of copper in the low-frequency limit, $\omega\tau \ll 1$. What is the value of δ_x for an electromagnetic field oscillating at frequency $f = 100\ \mathrm{GHz}$? How does the value of surface resistance $R_s = 1/(\sigma_0\delta_x)$ vary in the frequency range $1\ \mathrm{GHz} < f < 1\ \mathrm{THz}$?

(b) Find an expression for the skin depth of copper in the frequency range $f_\tau < f < f_p$. What is the value of δ_x at frequency $f = 100\ \mathrm{THz}$ and how does it vary with frequency in the range $10\ \mathrm{THz} < f < 1000\ \mathrm{THz}$?

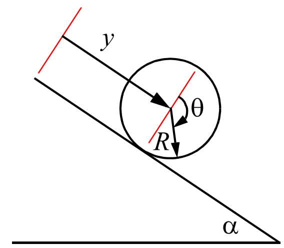
(c) What happens to the propagation of electric field when $f > f_p$?

Problem 1.20

(a) A uniform disk of mass m, initially at rest and with point-of-contact on a frictionless surface at position A, moves down the indicated surface in the presence of acceleration due to gravity. How long does it take the point of contact to reach point B? What value of h/L minimizes the time of travel?



(b) Suppose there is just enough friction to ensures a uniform disk radius R and mass m rolls down an inclined plane set at angle α without slipping. Show that its acceleration is 2/3 of the value it would have if there were no friction.



(c) How far does the disk travel on the surface (a) in the presence of friction described in (b)?

Problem 1.21
In fixed Cartesian coordinates a single particle with position coordinates x_i $(i = 1, 2, 3)$ has kinetic energy that is only a function of the time derivative $\dot{x}_i = dx_i/dt$ giving a Lagrangian defined as the difference between kinetic energy, $T = T(\dot{x}_i)$, and potential energy, $V = V(x_i)$ so that

$$\mathcal{L}(x_i, \dot{x}_i) = T - V \tag{1.1}$$

Hamilton's principle that states the path followed by a dynamical system moving from one point to another in configuration space within a given time interval, t_1 to t_2, minimizes the time integral of the Lagrangian. In the formalism of calculus of variations an extremum exists if the action integral

$$\delta\int_{t_1}^{t_2} \mathcal{L}(x_i, \dot{x}_i)dt = 0 \tag{1.2}$$

Eqn. (1.2) finds stationary action. Show, using calculus of variations, that the Lagrange equations of motion

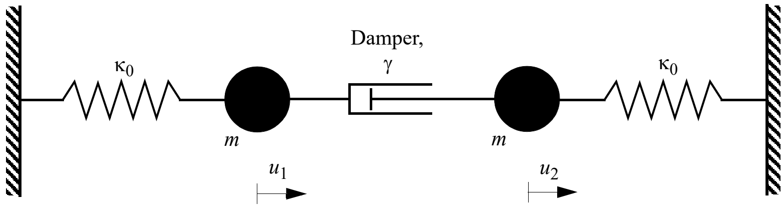
$$\frac{\partial\mathcal{L}}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} = 0 \tag{1.3}$$

may be obtained from the extremum of the action integral (Eqn. (1.2)) describing a dynamical system moving from one point to another in configuration space within a given time interval, t_1 to t_2.

Problem 1.22
A conservative system of N particles and $n \le 3N$ degrees of freedom has particle positions $x_i = x_i(q_1, q_2, \ldots, q_n)$ and $i = 1, 2, \ldots, 3N$ where the q_i form a proper set of independent generalized coordinates. Show that

$$\frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = 0$$

Problem 1.23
Two identical oscillators with motion in one-dimension, each of mass m and spring constant κ_0, are coupled by a damping piston with velocity-dependent friction force $-\gamma m(du_1/dt - du_2/dt)$, where γ is a damping rate and $u_{1,2}$ is particle displacement from equilibrium.



(a) Find the equations of motion and the eigenfrequencies for displacements u_1 and u_2.

(b) Solve for transient evolution of particle position when $\kappa_0/m = 1$, $\gamma = 0.1$, and the initial condition is $u_1 = 2$ and $u_2 = 1$ at time $t = 0$.

(c) Explain the motion remaining in the system in the limit $t \to \infty$. What percentage of initial energy is dissipated in the limit $t \to \infty$?

Problem 1.24

The standard forward finite-difference first derivative of a smooth real-valued function of real variable x is

$$f'(x) = \frac{f(x+h_0) - f(x)}{h_0}$$

This loses accuracy due to machine rounding error as h_0 becomes small. If u is the unit roundoff (e.g. $u \sim 10^{-16}$ for double precision) then $h_0 \sim \sqrt{u}$ minimizes both truncation error and rounding error. This difficulty using the finite difference approximation may be avoided by considering the analytic function of complex variable z that is real on the real axis with $f(z) = f(x + ih_0)$.

(a) For small h_0 expand $f(x + ih_0)$ in a Taylor series about x and show that

$$f'(x) = \mathrm{Im}\left(\frac{f(x+ih_0)}{h_0}\right) + E_1$$

where E_1 is the truncation error. Find the expression for E_1.

(b) Use the same approach to find the second derivative, $f''(x)$, and truncation error, E_2.

(c) For the function $f(x) = e^x/\sqrt{\cos^3(x) + \sin^3(x)}$ compute $f'(x = \pi/4)$ by the complex method and the finite difference method. Use MATLAB's Symbolic toolbox to find the actual derivative $f'(x = \pi/4)$. Compare and plot the errors in both methods for the range $10^{-1} < h_0 < 10^{-16}$.

Problem 1.25

The Shannon entropy is conventionally defined for a one-dimensional (1D) discrete probability distribution $\{p_1, \ldots, p_N\}$ of a single variable x (where $p_i = p(x_i)$ and $\sum_i^N p(x_i) = 1$) as

$$H(x) = H(p_1, \ldots, p_n) \equiv -\sum_{i=1}^{N} p(x_i)\log_2(p(x_i))$$

in which log base two is used to indicate entropy units of bits. Furthermore, for the case where the probability distribution contains zeros, $\lim_{x\to 0} x\log_2(x) = 0$.

(a) Consider a system with a binary outcome, such as a conventional coin toss. Use MATLAB to plot the Shannon entropy as a function of p_1 (the probability of measuring heads, for example; the probability of measuring tails is automatically constrained by the sum rule). Is this function convex, concave, or monotonic? How many bits is the maximum entropy, and for what value(s) of probability does this value occur? Prove this mathematically. What is the interpretation of a bit in this context?

(b) Now consider a system with three possible outcomes. Use MATLAB to generate a surface plot of the Shannon entropy as a function of p_1 and p_2. Numerically determine the maximum value of entropy in bits, and for what values of p_1 and p_2 (and thus p_3) does this entropy occur? Prove this mathematically. Comment on any similar features between the binary entropy curve and this ternary entropy surface.

(c) In general, what type of probability distribution maximizes the entropy for these systems with discrete numbers of states, and how does the maximum entropy scale with respect to an N-outcome system with this type of distribution? What does this say about obtaining information through measurement and the allowed amount of information that can be stored?

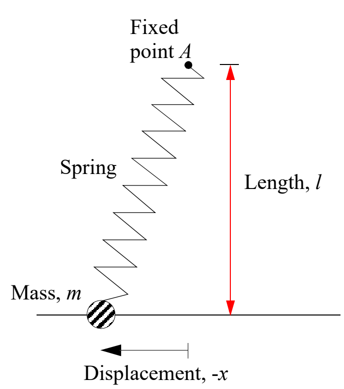
Problem 1.26

A stationary ground-based radar uses a continuous electromagnetic wave at 10 GHz frequency to measure the speed of a passing airplane moving at a constant altitude and in a straight line at 1000 km hr^{-1}. What is the maximum beat frequency between the out going and reflected radar beams? Sketch how the beat frequency varies as a function of time. What happens to the beat frequency if the airplane moves in an arc?

Problem 1.27

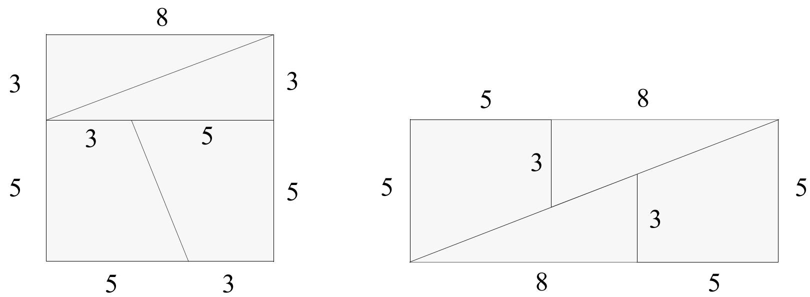
(a) Find the frequency of oscillation of the particle of mass m illustrated in the figure below. The particle is only free to move along a line and is attached to a light spring whose other end is fixed at point A located a distance l perpendicular to the line. A force F_0 is required to extend the spring to length l.

(b) Part (a) describes a new type of child's swing. If the child weighs 20 kg, the length $l = 2.5$ m, and the force $F_0 = 450$ N, what is the period of oscillation?



Problem 1.28

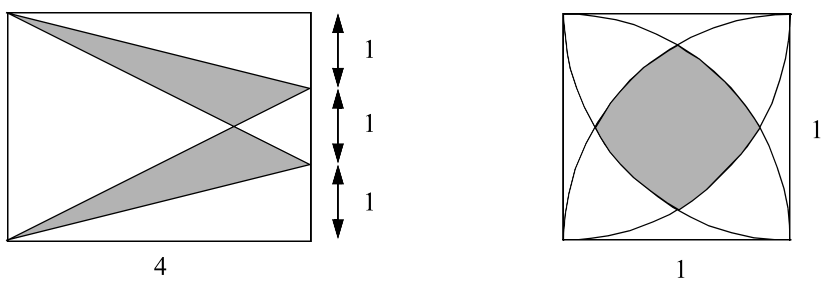
(a) Explain the discrepancy in the following:



64 square units   65 square units

(b) Calculate the shaded area of the following symmetric designs labeled (i) and (ii). The square in (ii) has side 1 and the arcs have radius 1.

(i)   (ii)



Problem 1.29

Analytic solutions of a damped harmonic oscillator of natural frequency ω_0 with motion constrained to the x-direction and subject to a small external oscillatory force, $F(t) = F_0 e^{i\omega t}$ are known.

(a) Assuming the force is applied for time $t \geq 0$, solve the problem numerically by directly integrating the response of the system in time. Plot $x(t)$ to show the transient and steady-state behavior. Plot and explain the behavior of $\frac{\mathrm{d}}{\mathrm{d}t}x(x) \equiv \dot{x}(x)$ when the system has reached steady-state. What is the time evolution of the total energy of the system when ω is detuned from ω_0? To ensure accuracy use the fourth-order Runge-Kutta method and implement your code in MATLAB.

(b) Add a non-linear term such that the restoring force is of the form $F = -\kappa_1 x - \kappa_3 x^3$. Plot $x(t)$, $\dot{x}(x)$, and the potential $V(x)$ for both positive and negative values of κ_1. Explain the different types of motion visualized by $x(t)$ and $\dot{x}(x)$ that you are able to achieve by varying the potential and driving force.

Problem 1.30

Starting from the characteristic polynomial describing motion in a diatomic linear chain

$$\omega^4 - 2\kappa\left(\frac{m_1 + m_2}{m_1 m_2}\right)\omega^2 + \frac{2\kappa^2}{m_1 m_2}(1 - \cos(qL)) = 0$$

where L is the unit-cell lattice constant, $m_{1,2}$ the atom mass, and κ the spring constant. Consider the limit $q \to 0$, expand the term $\cos(qL) = 1 - q^2L^2/2 + \ldots$ and find an expression for the group velocity, $\upsilon_g = \partial\omega/\partial q$, of acoustic sound waves.

Assuming this model may be used to approximate the vibrational properties of a silicon crystal with lattice constant $L_{Si} = 0.5431$ nm, atom mass $m = 4.7 \times 10^{-26}$ kg, and room-temperature sound velocity of $\upsilon_g = 8.4 \times 10^3$ m s^{-1} in the (100)-direction, find the value of κ_{Si} and predict the maximum frequency of oscillation of the optic branch in the dispersion relation. Compare with the experimentally measured result for silicon.

Problem 1.31

Gramse et al. (*Sci. Adv*. **3**, e1602586, 28 June 2017 (DOI: 10.1126/sciadv.1602586)) show that deep sub-wavelength imaging is possible using 20 GHz RF radiation ($\lambda_0 = 1.5$ cm) emitted from a metal tip scanned at the surface of the structure to be probed. Sub-40 nm lateral and 4 nm ($\sim \lambda_0/4 \times 10^6$) vertical features are resolved.

Explain why the experiment works and its limitations.