Applied Quantum Mechanics

Chapter 2 problems

LAST NAME FIRST NAME

Problem 2.1

(a) The Sun has a surface temperature of 5800 K and an average radius 6.96×10^8 m. Assuming the mean Sun-Mars distance is 2.28×10^{11} m, what is the total radiative power per unit area incident on the upper Mars atmosphere facing the Sun?

(b) If the surface temperature of the Sun was 6800 K, by how much would the total radiative power per unit area incident on Mars increase?

Problem 2.2

(a) A positron has charge $+e$ and the same mass m_0 as a bare electron. The energy of a particle with rest mass m_0 moving at velocity υ with momentum $p = \gamma_{\text{Lorentz}} m_0 \upsilon$ is $E = \gamma_{\text{Lorentz}} m_0 c^2$ where c is the speed of light and γ_{Lorentz} is the Lorentz factor such that $\gamma_{\text{Lorentz}}^2 = c^2/(c^2 - \upsilon^2)$. Why does a single high-energy photon *not* decay into an electron and a positron?

(b) Experiments show that *two* colliding real photons γ_1 and γ_2 can create particles that have mass (D. L. Burke et al. *Phys. Rev. Lett.* **79**, 1626 (1997)). Describe the conditions in which these photons decay into a positron and an electron.

Problem 2.3

Consider a lithium atom (Li) with two electrons missing.

(a) Draw an energy level diagram for the Li^{++} ion.

(b) Derive the expression for the energy (in eV) and wavelength (in nm) of emitted light from transitions between energy levels.

(c) Calculate the three longest wavelengths (in nm) for transitions terminating at $n = 2$.

(d) If the lithium ion were embedded in a dielectric with relative permittivity $\varepsilon_r = 10$, what would be the expression for the energy (in eV) and wavelength (in nm) of emitted light from transitions between energy levels.

Problem 2.4

A particle mass m moving in a real potential is described by wave function $\psi(x, t)$.

(a) Write down the expression for the average value of the particle position $\langle x \rangle$ and then make use of the Schrödinger equation to show that the average value of momentum is

$$\langle p \rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\langle x \rangle = \frac{\mathrm{i}\hbar}{2}\int_{-\infty}^{\infty} x\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}x} - \frac{\mathrm{d}\psi^*}{\mathrm{d}x}\psi\right)\mathrm{d}x$$

(b) Evaluate the integral in part (a) and show that

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*\left(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi\,\mathrm{d}x$$

so that one may identify the momentum operator as

$$\hat{p} = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$$

Problem 2.5

Create a simple model of a heterostructure diode that predicts current increases exponentially with increasing forward voltage bias. Explain the assumptions you make to develop the model. Under what conditions will this predicted behavior fail? By how much is voltage bias across an ideal diode increased to change current by a factor of 10 at room temperature (T = 300 K)? Does this represent a fundamental limit to power dissipation in electronic switching devices operating at room temperature?

Problem 2.6

Write down the Hamiltonian operator for (a) a one-dimensional simple harmonic oscillator, (b) a helium atom, (c) a hydrogen molecule, (d) a molecule with n_n nuclei and n_e electrons.

Problem 2.7

Calculate the classical velocity of the electron in the n-th orbit of a Li^{++} ion. If this electron is described as a wave packet and its position is known to an accuracy of Δx = 1 pm, calculate the characteristic time $\Delta\tau_{\Delta x}$ for the width of the wave packet to double. Compare $\Delta\tau_{\Delta x}$ with the time to complete one classical orbit. Should the electron be described as a particle or a wave?

Problem 2.8

What is the Bohr radius for an electron with effective electron mass $m^* = 0.021 \times m_0$ in a medium with low-frequency relative permittivity $\varepsilon_{r0} = 14.55$ corresponding to the conduction band properties of single crystal InAs? What is the effective Rydberg energy for an electron describing a hydrogenic orbit in the medium?

Problem 2.9

Because electromagnetic radiation possesses momentum it can exert a force. If completely absorbed by matter, the absorbed electromagnetic radiation energy per unit time per unit area is a pressure called radiation pressure.

(a) If the maximum radiative power per unit area incident on the upper Earth atmosphere facing the Sun is 1.4 kW m^{-2}, what is the corresponding radiation pressure?

(b) Estimate the photon flux needed to create the pressure in (a).

(c) Compare the result in (a) with the pressure due to one atmosphere.

(d) Assuming Poisson statistics applies to part (b), what is the fluctuation in pressure per unit time?

Problem 2.10

(a) As described in Chapter 2, Alice can transmit information to Bob via a quantum communication channel that uses single photons and nonorthogonal polarization states. Explain Bob's choice of test basis in Fig. 2.8.

(b) In the absence of a single photon source, optical quantum key distribution (QKD) uses light from an attenuated laser. In a particular system the mean photon number per pulse is 0.1 and the probability of single photon emission is 0.09. The link operates with a clock rate of 1.25 GHz (bit time τ = 800 ps), average optical loss in the link is -10 dB, and time jitter in the photodetector requires two bit-time intervals be used for photon detection. What is the maximum sustained data rate for guaranteed secure QKD in the system?

(c) No light can pass between two linear polarizers if their respective polarizations are oriented at 90° to each other. If a third linear polarizer oriented at a 45° angle is placed between the two linear polarizers, what is the maximum fraction of incident light intensity that can pass through the system?

Problem 2.11

A particle mass m moving in a real potential is described by wave function $\psi(x, t)$ and Schrödinger's equation. Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 0$$

so that if the wave function $\psi(x, t)$ is normalized it remains so for all time.

Problem 2.12

Fixed positive charge is uniformly distributed in the volume of a sphere radius R with charge density $\rho = e \times 1.5 \times 10^{28}\ \mathrm{m}^{-3}$.

(a) Calculate the force on a electron point-particle initially placed at the surface of the sphere.

(b) Assuming the electron in (a) is free to move in the volume of the sphere, what is the potential seen by the electron and what is its subsequent motion?

(c) What energy and wavelength of photons do you anticipate being absorbed by the system?

(d) At what radius R_c does your model predict the velocity of the electron exceeds the speed of light in vacuum?

(e) If the total system consisting of the sphere of positive charge density and the electron is charge neutral what is the value of the radius, R? How does electron dispersion limit the validity of your model?

Problem 2.13

The wave function at time $t = 0$ for an electron localized as a Gaussian wave packet in one-dimension centered at $x = x_0$ and having spatial width Δx and momentum $\hbar k_0$ is

$$\psi(x, t = 0) = Ae^{ik_0 x}e^{-(x - x_0)^2/(4\Delta x^2)}.$$

(a) Find the wave function normalization constant A and show that $\langle x \rangle = x_0$ and $\sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \Delta x$.

(b) Take the Fourier transform of $\psi(x, t = 0)$ to determine $\psi(k, t = 0)$. Making the substitution $\Delta k = (2\Delta x)^{-1}$, compare your result to that of part (a). Explain the significance of $\Delta k \Delta x = 1/2$.

Hint: You may use Cauchy's integral theorem, which states that $\oint f(z)dz = 0$ where z is a complex variable and $f(z)$ has no poles within the closed integration loop.

(c) Find $\psi(x, t = 0)$ and $\psi(k, t = 0)$ in the limit $\Delta x \to 0$ and describe how the electron wave packet evolves with time. If we were to measure the electron's location with absolute certainty at time $t = 0$, where can we expect to locate the electron at any subsequent time? Explain your result.

(d) *If* a localized single *photon* in free-space can be described by the same initial Gaussian wave packet $\psi(x, t = 0)$, how will it evolve in time? Explain the difference in the predicted behavior of the photon compared to the electron.

Problem 2.14

Optical quantum key distribution (QKD) protocol may be viewed as a sensor capable of detecting eavesdropping on an optic link. Alice and Bob use the B92 protocol and measure error rate to detect the presence of Eve, an eavesdropper, in an otherwise lossless optical QKD link.

(a) What is the average error ratio generated by eavesdropping?

(b) What can Alice and Bob infer about the methods used by Eve?

Problem 2.15

In quantum mechanics the $\pm\hbar/2$ spin of an electron charge e and mass m_0 emerges as a consequence of rotational symmetry and a relativistic treatment of the Schrödinger equation.

(a) *Suppose* spin were a classical concept due to the rotation of a spherical electron of classical radius $r_e = \dfrac{e^2}{4\pi\varepsilon_0 m_0 c^2}$ about an axis passing through its center. Assuming a uniform classical electron density in the sphere, explicitly find the moment of inertia of such an electron.

(b) If the angular momentum of the classical electron is $\hbar/2$ what is the speed of the surface of the electron at its equator? Compare this to the speed of light, c, and comment on what you have learned.

(c) If all the electron density is in a thin torus of vanishingly small cross-section making a ring at the equator calculate the radius at which the ring speed is 1% of c. Calculate the current in the ring and the magnetic field through the ring.

(d) Calculate the magnitude and direction of the classical force due to the coulomb interaction experienced by the electron charge density of the ring in (c).

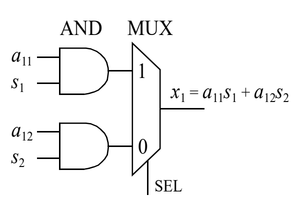
(e) What happens to the force you calculated in (d) if, as in quantum mechanics, the electron is a point particle?

Problem 2.16

Random generation of binary 1 or 0 can be physically guaranteed by quantum mechanics and so there is interest in using this as a resource for stochastic computing. Numbers can be represented as probabilities of a binary 1 or 0 signal in a clocked bit-stream of length n_{bits} such that as $n_{\text{bits}} \to \infty$ the average value of the signal is a number distributed in the interval [0,1]. For a 2×2 matrix, $\mathbf{x} = \mathbf{A}\mathbf{s}$ may be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

so that evaluation of $x_1 = a_{11} \times s_1 + a_{12} \times s_2$ requires both multiplication and addition. A circuit that can perform this multiply-accumulate function on clocked stochastic data is



where the select (SEL) is random and uniformly distributed in the interval [0,1] and the output is scaled to be in the interval [0,1].

(a) When evaluating the determinant $|\mathbf{A}|$ how does root-mean-square (RMS) error scale as a function of n_{bits} and two-norm condition number of a poorly conditioned matrix?

(b) Multiplication of two stochastic bit-streams can be achieved using the AND operation and addition can be achieved using the MUX function with a random select. What circuit elements can perform (i) subtraction and (ii) division of two stochastic bit-streams?

Problem 2.17

The dispersion of an electron initially localized as a Gaussian wave packet was studied in problem 2.13. However, the dispersion was only qualitatively described as spread in width of the pulse because a proper *measure* of pulse dispersion was not defined.

(a) Consider a Gaussian pulse describing an electron with effective mass m_e^* and average energy E moving in the x-direction at velocity $\mathbf{u}$ and average momentum $\mathbf{p} = m_e^*\mathbf{u}$. The standard deviation of the pulse about $x = 0$ at time $t = 0$ is b so that the wave function is

$$\psi(x,t) = \frac{b}{\sqrt{b^2 + \frac{i\hbar t}{m_e^*}}}\, e^{\,i\frac{m_e^* u}{\hbar}\left(\left(x - \frac{ut}{2}\right)\right) - \frac{(x-ut)^2}{2(b^2 + i\hbar t/m_e^*)}}$$

Find the overlap area $D(t)$ of $|\psi(x,t)|^2$ with $|\psi(x,0)|^2$ when mean position of $|\psi(x,t)|^2$ is shifted to $x = 0$, for time in the range $0 \le t \le 0.6$ ps. Obtain the quantitative measure of pulse dispersion $D(t)$ using parameters $E = 50$ meV, $m_e^* = 0.07 \times m_0$ and $b = 10$ nm.

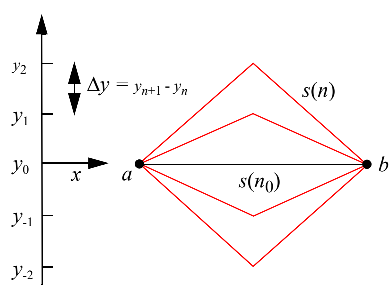
(b) Show that the pulse dispersion measure is bounded to $0 \le D \le 1$

Problem 2.18

The shortest path length of a particle with wave properties moving in free-space in the x-direction from point a to point b is $s(n_0)$. The particle has energy $E = \hbar\omega$ and momentum $\hbar\mathbf{k}$. Assuming that at any instant $\mathbf{k}$ is parallel to the path taken $s(n)$ then, since in free-space $A_n = A_0$, the total amplitude at b is the sum over all possible paths

$$\mathbf{A}_{\text{tot}} = \sum_n A_n e^{iks(n) - i\omega t} = A_0 e^{-i\omega t}\sum_n e^{iks(n)} = A_0 e^{-i\omega t}\sum_{n=-n_{\max}}^{n=n_{\max}} e^{i\phi(n)}$$

where $\phi(n)$ is the phase at point b for the path $s(n)$.



For the equilateral triangle paths with constant step increase in midpoint height Δy shown in the figure, plot $\mathrm{Im}(\phi)$ as a function of $\mathrm{Re}(\phi)$ using parameters $s(n_0) = 1$, $k = \pi$, $\Delta y = 0.005$, and $n_{\max} = 2000$. Explain the results you obtain.

Problem 2.19

Identical spherical atoms of unit density are arranged in space so that the nearest neighbor atoms just touch. Show that such a close-packed arrangement results in density (or "packing fraction") for the indicated cubic lattices:

Structure	Density
Diamond	$\sqrt{3}\frac{\pi}{16} = 0.34$
Simple cubic (SC)	$\frac{\pi}{6} = 0.52$
Body-center cubic (BCC)	$\sqrt{3}\frac{\pi}{8} = 0.68$
Face-center cubic (FCC)	$\sqrt{2}\frac{\pi}{6} = 0.74$

Problem 2.20

The intensity of two co-propagating electromagnetic waves with identical frequency f_0, polarization, and wavelength λ_0 depends on their phase difference.

(a) Write a MATLAB program that reproduces Fig. 2.2 using two waves of frequency $f_0 = 200\text{ THz}$. Plot the normalized electric field intensity versus phase delay. Consider phase delays up to $\pm 4\lambda_0$.

(b) Modify your code from part (a) so that each wave consists of two equal amplitude component waves, one with frequency $f_1 = 220\text{ THz}$ and the other with frequency $f_2 = 180\text{ THz}$. Plot the normalized field intensity versus phase delay. Consider phase delays up to $\pm 8\lambda_0$.

(c) Modify your code from part (b) so that the component wave amplitude has a Gaussian distribution as a function of frequency. The Gaussian distribution has a standard deviation of 20 THz about the center frequency, $f_0 = 200\text{ THz}$. Plot the normalized field intensity versus phase delay for up to $\pm 8\lambda_0$ when each wave consists of 3, 4, 5, and 10 component frequencies evenly spaced between ± 3 standard deviations. What is the field intensity at a delay length of $8\lambda_0$ when 10 component frequencies are included in the calculation? Explain the reason for this value.

Include a printout of your final code from part (c) as well as the generated plots with your assignment.

Problem 2.21

The normalized first-order time-independent correlation function of a classical electric field $E(t)$ is $g^{(1)}(\tau) = \dfrac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}$, where the brackets $\langle\ \rangle$ indicate an average over all time, t.

(a) Show that $g^{(1)}(\tau) = g^{(1)}(-\tau)^*$, $|g^{(1)}(\tau)| \leq 1$, and that light of single frequency ω_0 has $g^{(1)}(\tau) = e^{-i\omega_0\tau}$.

(b) The probability there are n photons in a single radiation mode of a cavity with frequency ω is given by Boltzmann's law

$$P_\omega(n) = \frac{e^{-E_n/k_B T}}{\sum_{n=0} e^{-E_n/k_B T}}$$

where the quantized energy of a photon is $E_n = (n + 1/2)\hbar\omega$. Show that $P_\omega(n)$ may be written as $P_\omega(n) = (1 - e^{-\hbar\omega/k_B T})e^{-n\hbar\omega/k_B T}$ and that the mean photon number is $\bar{n} = 1/(e^{\hbar\omega/k_B T} - 1)$. Show that $P_\omega(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$.

(c) The second-order time-independent correlation function of classical electric field intensity $I(t) = E^*(t)E(t)$ with a stationary statistical distribution is

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t)\rangle}$$

Show that $g^{(2)}(\tau) = g^{(2)}(-\tau)$, $g^{(2)}(\tau) \geq 0$, $g^{(2)}(0) \geq 1$, and $g^{(2)}(0) \geq g^{(2)}(\tau)$.

(d) Calculate $g^{(2)}(\tau = 0)$ for a monochromatic wave with sinusoidal intensity modulation of the form $I(t) = I_0(1 + A\sin(\omega_0 t))$, where $|A| \leq 1$ and I_0 and ω_0 are constant.

(e) Simulate the electric field $E(t)$ of thermal light by using MATLAB to generate and plot an array of at least 100,000 random Gaussian numbers with unit standard deviation and mean of zero. Write a program to spectrally filter $E(t)$ by convolving with a Gaussian function of standard deviation $\tau_C = 50$ data points to obtain $E_{\mathrm{Filter}}(t)$. Calculate and plot the intensity for 2000 data points along with a probability histogram of intensity for both $E(t)$ and $E_{\mathrm{Filter}}(t)$. Verify for both cases that the histogram follows the thermal distribution given by $P(I) = \frac{1}{I_0}e^{-I/I_0}$. Plot $g^{(1)}(\tau)$ and $g^{(2)}(\tau)$ using $E_{\mathrm{Filter}}(t)$ for time-delay interval range ± 500 data points. Verify $g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$ and that $g^{(1)}(\tau)$ is of the form $e^{-(\tau/\tau_C)^2}$. Explain your results.

Problem 2.22

As illustrated in the following figure, we start by assuming it is possible to model one aspect of the photoelectric effect as a single photon of energy $\hbar\omega$ colliding with an electron mass m_0 with initial energy E and momentum $\hbar\mathbf{k}$ whose trajectory is normal to that of the photon. After the collision the final momentum of the electron makes angle θ with its initial momentum.

(a) Suppose it is possible to transfer all the initial momentum of the photon to the electron. Use momentum conservation to find electron scattering angle, θ, as a function of photon energy, $\hbar\omega$, and, if initial electron energy is $E = 1$ eV and $\theta = \pi/4$, what is the value of photon energy, $\hbar\omega$?

(b) Explain why the scenario in (a) is not physical and how you would develop a correct model of the photoelectric effect.