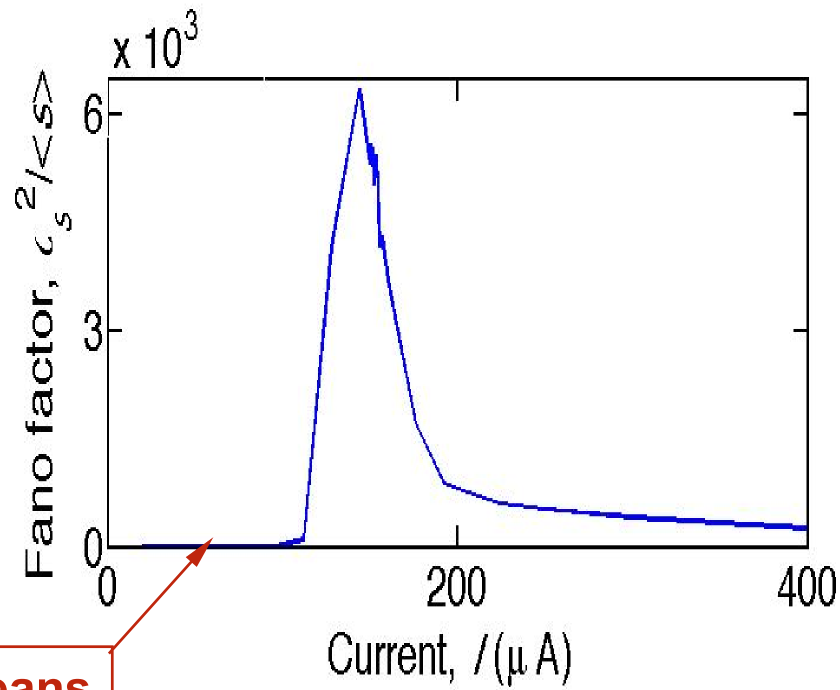
Characterizing fluctuations: The Fano-factor

- Fano-factor = $\sigma_s^2/\langle s\rangle = (\langle s^2\rangle - \langle s\rangle^2)/\langle s\rangle$ measures the strength in photon fluctuations
- Peaks sharply across threshold if a threshold exists and the system undergoes a phase transition
- For Poisson fluctuations $\sigma_s^2/\langle s\rangle = 1$



Fano factor = 1, means Poisson fluctuations

Previous work – quantum theory

- In a full quantum statistical approach both the medium and light is quantized
- This treatment of lasers can be broadly classified into two categories:
  - A phase-space description in terms of Fokker-Plank equations by Haken, which accounts for the small fluctuations in the system about the mean
    - Only calculated for large systems with small fluctuations
  - The density matrix description of Scully and Lamb
    - Only calculated for large systems with small fluctuations
- Theories are complete in the sense that they provide information about particle statistics and laser line width, but, in practice, they have only been solved for large systems

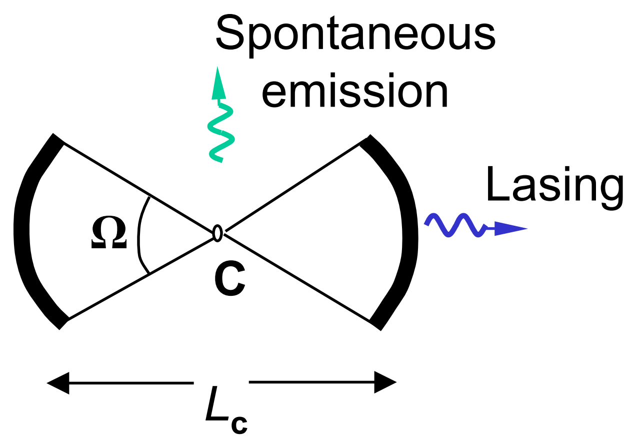
Ask a simple question:

How do small laser diodes behave?

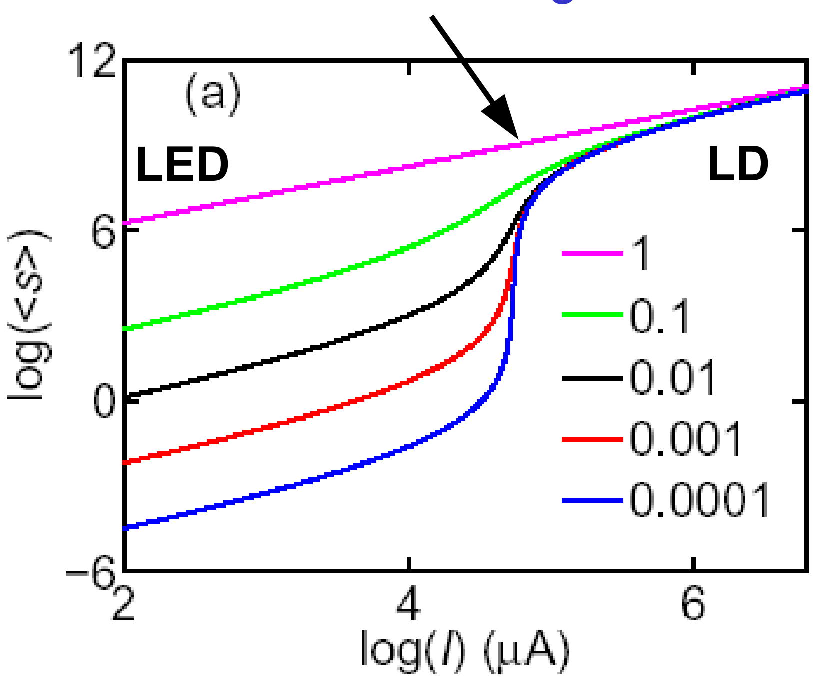
What happens when a laser is made small?

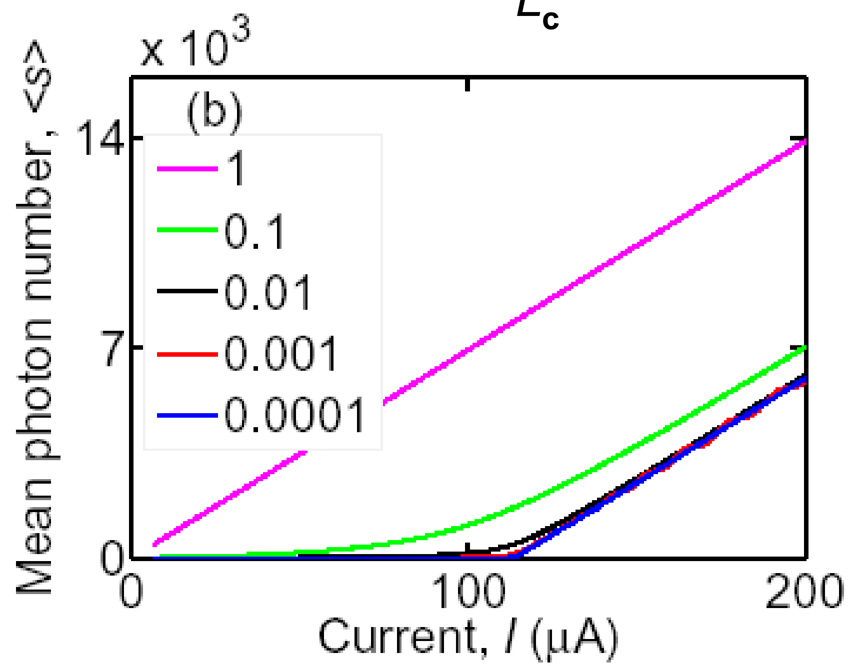
- More spontaneous emission into lasing mode, $\beta \sim 0.1$
- Continuum mean-field rate equations predict “soft” threshold
  - Active volume 0.12 $\times 10^{-12}$ cm^{-3} = 0.12 μm^3
  - $I_{th} < 1$ mA, $\langle n \rangle = 2 \times 10^5$, $\langle s \rangle = 10^3$

$\beta = \Omega/4\pi$ ➡ $L_c\,(\longrightarrow 0)$, $\Omega(\longrightarrow 4\pi)$, $\beta\,(\longrightarrow 1)$



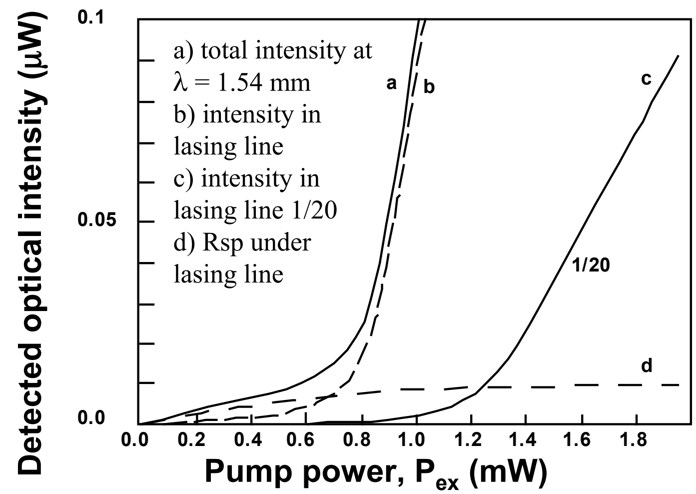
“Thresholdless” lasing



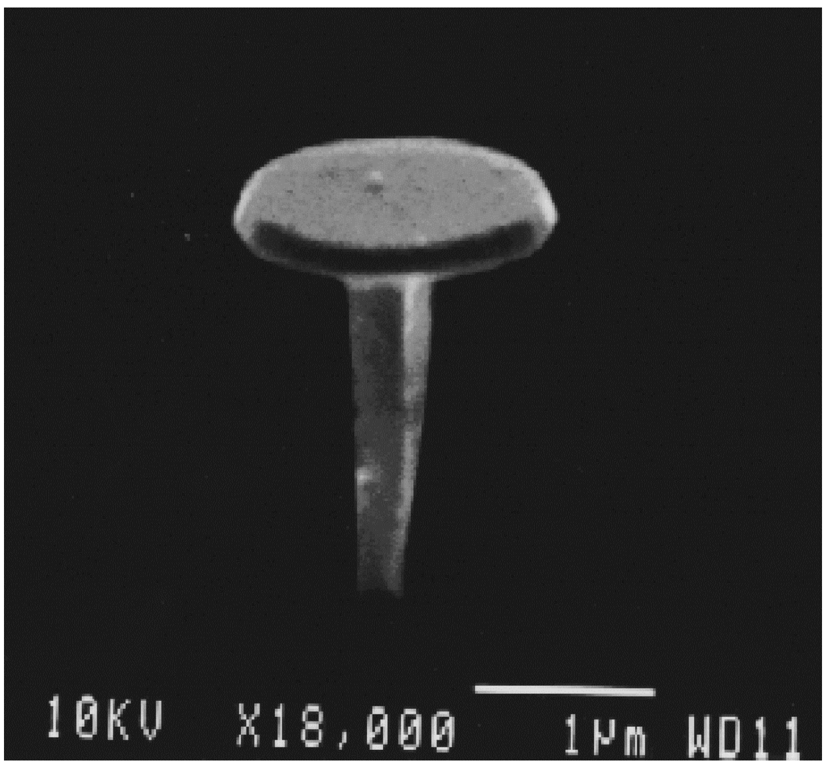


What happens when a laser is made small?

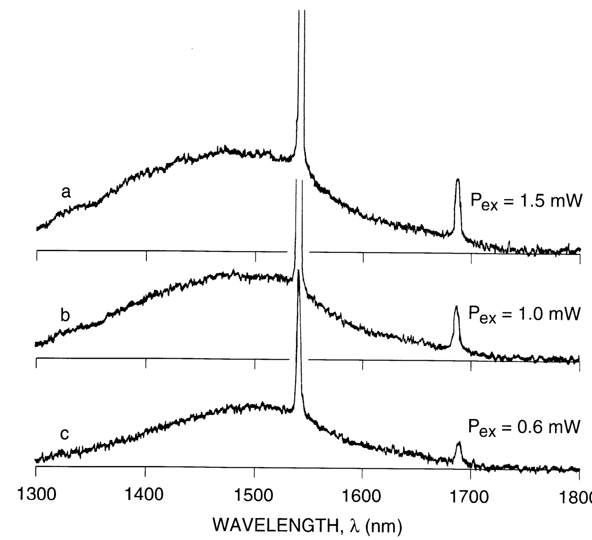
- More spontaneous emission into lasing mode, $\beta \sim 0.1$
- Fluctuations more important
  - Active volume 0.12×10^{-12} cm^{-3}
  - $I_{th} < 1$ mA, $\langle n \rangle = 2 \times 10^5$, $\langle s \rangle = 10^3$



Optically pumped 6-QWs microdisk laser input-output characteristics



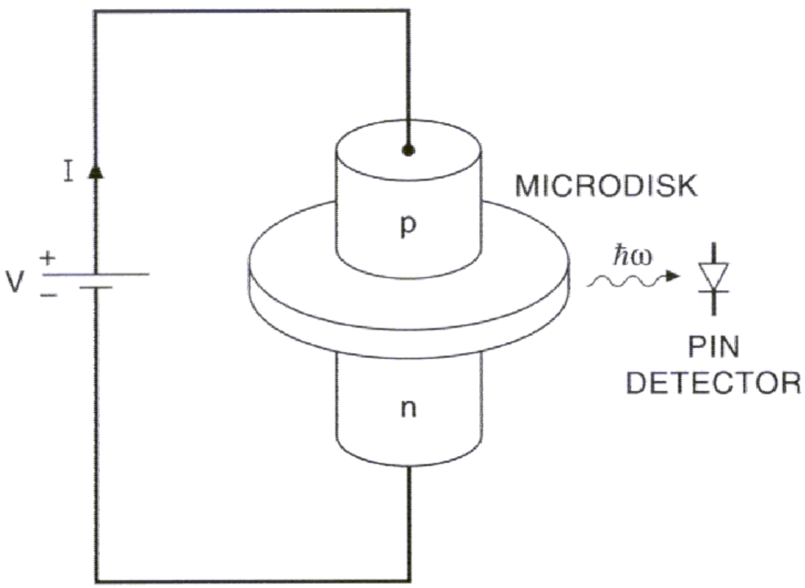
6-QW microdisk laser, R = 0.8 μm, h = 0.18 μm

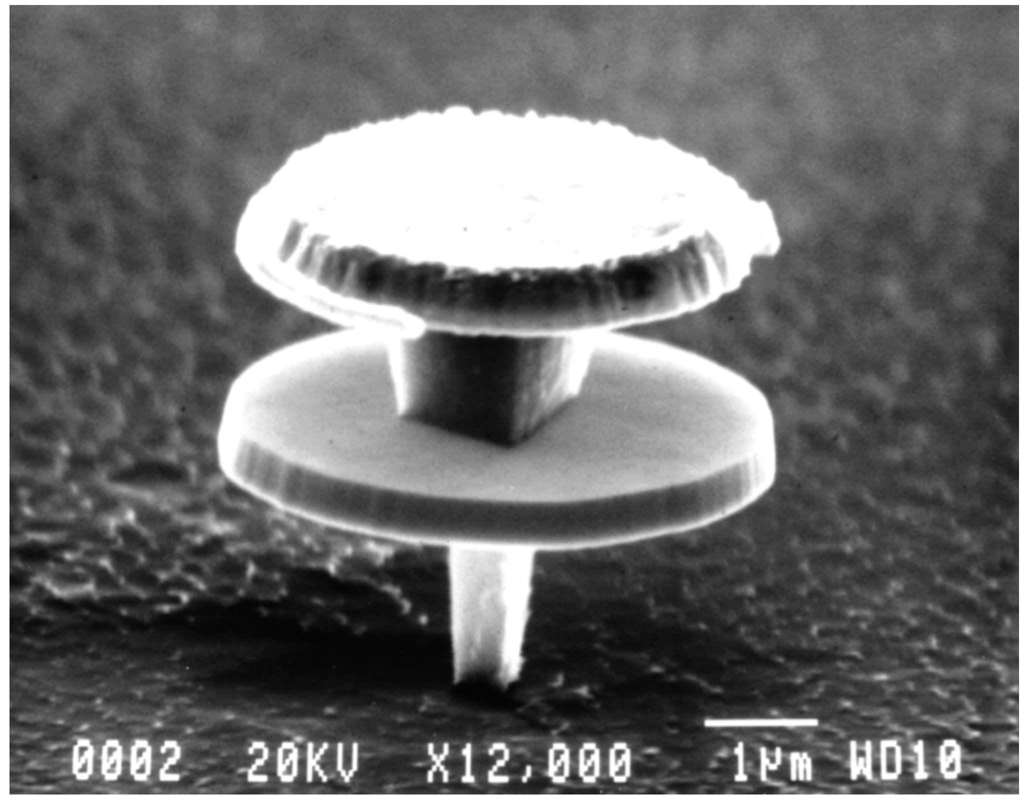


Room temperature emission spectra mode spacing $\Delta\lambda$ = 1690-1542 = 148 nm

Electrically driven microdisk laser diode

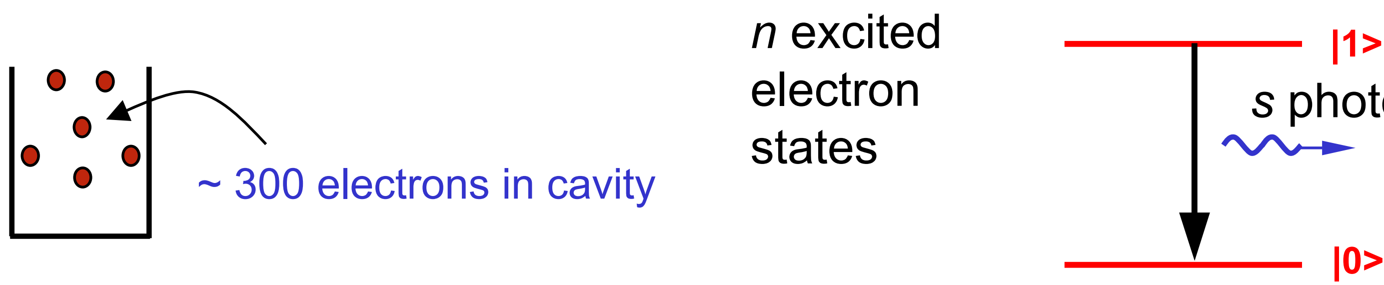
- Small active and cavity volume





Quantum fluctuations in a *very* small laser

- Small active and cavity volume
  - Small electron number, *n*, and photon number, *s*



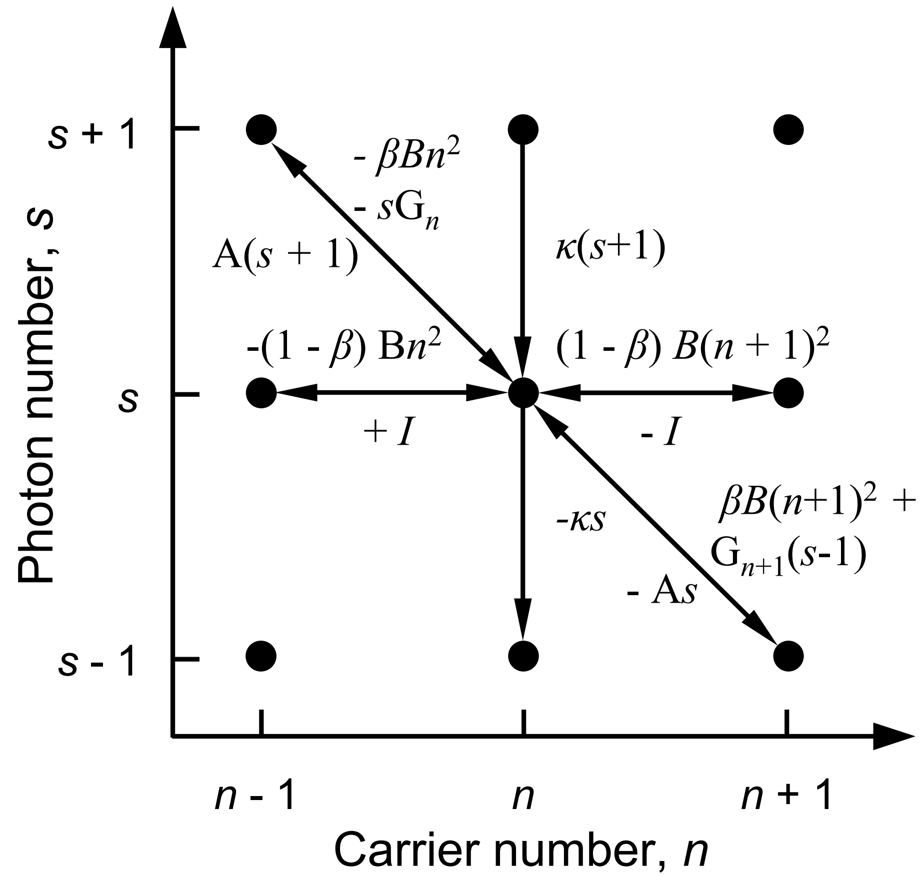
- Fluctuations ($\langle s^2 \rangle$ and $\langle n^2 \rangle$) and correlations ($\langle ns \rangle$) in particle number become important
- Mean-field theories (rate equations and Gaussian noise - Langevin) can not be used
- What effect do large fluctuations and strong correlations have on steady–state and temporal behavior of small lasers in the quantum regime?

Quantum fluctuations in a *very* small laser

- Solving the full quantum mechanical problem is difficult at the meso-scale because the number of system states can become very large
  - Small electron number, n (~300), and small photon number, s (~100)
  - The number of system states scales as $(2^n) \times s$, where n is the number of two-level electronic states and s is the number of photons in the lasing mode
  - The corresponding coefficient matrix has size $(2^n)s \times (2^n)s$ and so the problem becomes computationally challenging with increasing number of electronic states inside the cavity
  - For only 20 two-level electronic states and 100 photons $(2^n) \times s$ is state vector of length about 10^8 (note $(2^{300}) \times 100 \sim 2 \times 10^{92}$ which is more than the number of atoms in the universe)
- Need a different approach

Master equations

- First-cut at capturing quantum effects
  - Photon energy $\hbar\omega$
  - Quantize photon number *s* and excited electron particle number *n*
  - Weak coupling
- Use master equations (a set of differential equations in continuous probability functions, P_{ns}) to describe the system



$$d\langle n\rangle/dt = I - B\langle n^2\rangle - a\Gamma\langle n-n_0\rangle\langle s\rangle/V$$

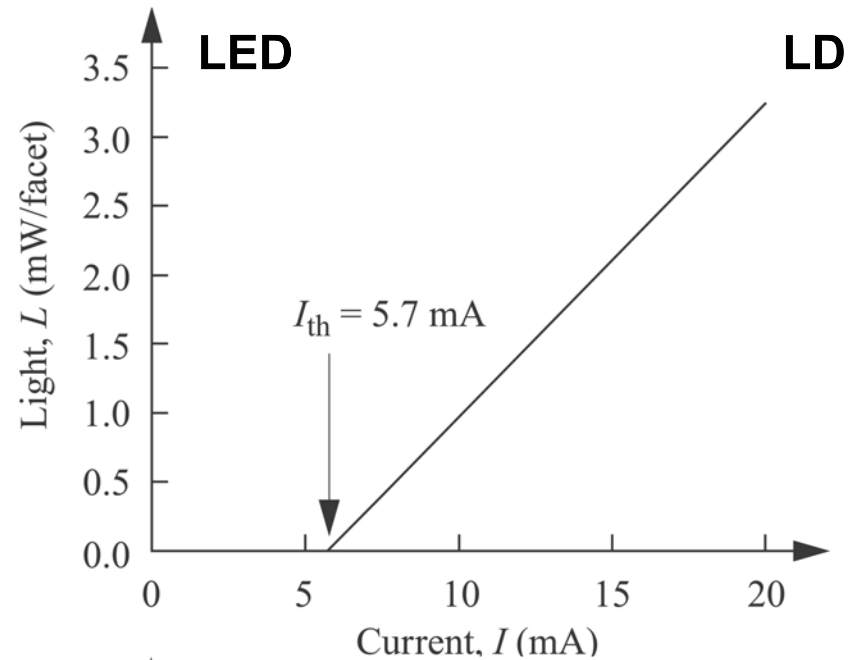
$$d\langle s\rangle/dt = \beta B\langle n^2\rangle + a\Gamma\langle n-n_0\rangle\langle s\rangle/V - \kappa\langle s\rangle$$

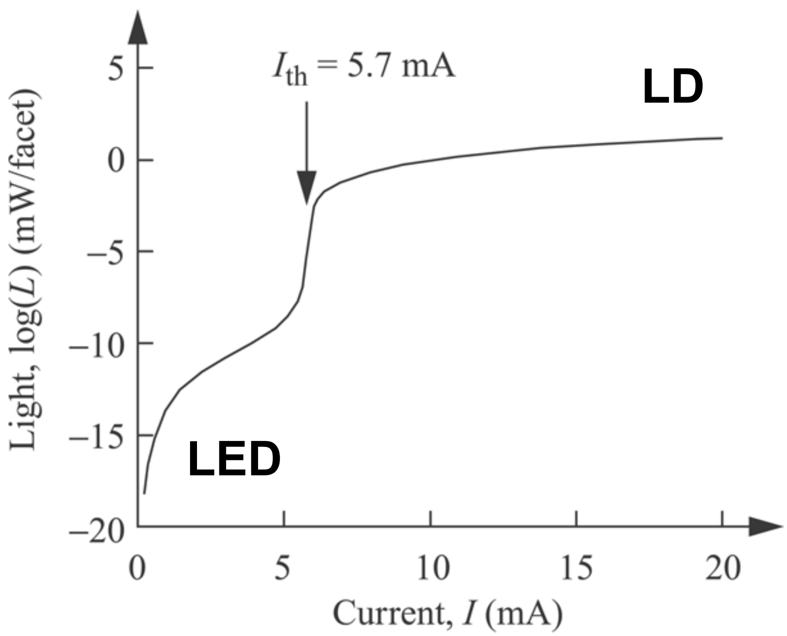
Continuum mean-field rate equations set $\langle ns\rangle = \langle n\rangle\langle s\rangle$
Master equations , $\langle ns\rangle \neq \langle n\rangle\langle s\rangle$

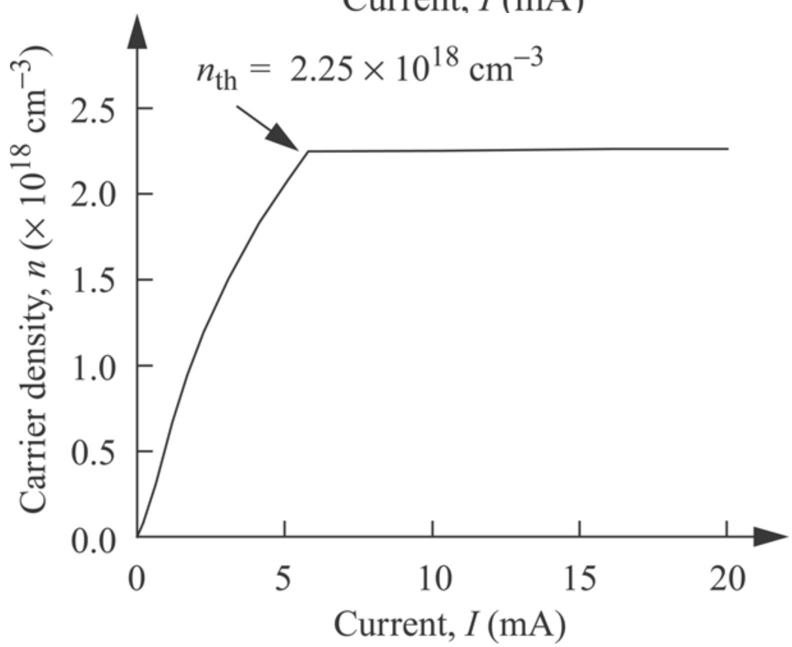
$$\frac{dP_{n,s}}{dt} = -\kappa(sP_{n,s} - (s+1)P_{n,s+1}) - (sG_nP_{n,s} - (s-1)G_{n+1}P_{n+1,s-1}) - (sAP_{n,s} - (s+1)AP_{n-1,s+1})$$
$$- \beta B(n^2P_{n,s} - (n+1)^2P_{n+1,s-1}) - (1-\beta)B(n^2P_{n,s} - (n+1)^2P_{n+1,s}) - I(P_{n,s} - P_{n-1,s})$$

Continuum mean-field rate equation prediction

- Steady-state behavior predicted by continuum mean-field rate equations
  - Threshold current I_{th}
  - Carrier number n pinned when $I > I_{th}$







Continuum mean-field rate equation prediction

- Why carriers are pinned above I_{th}

In steady-state

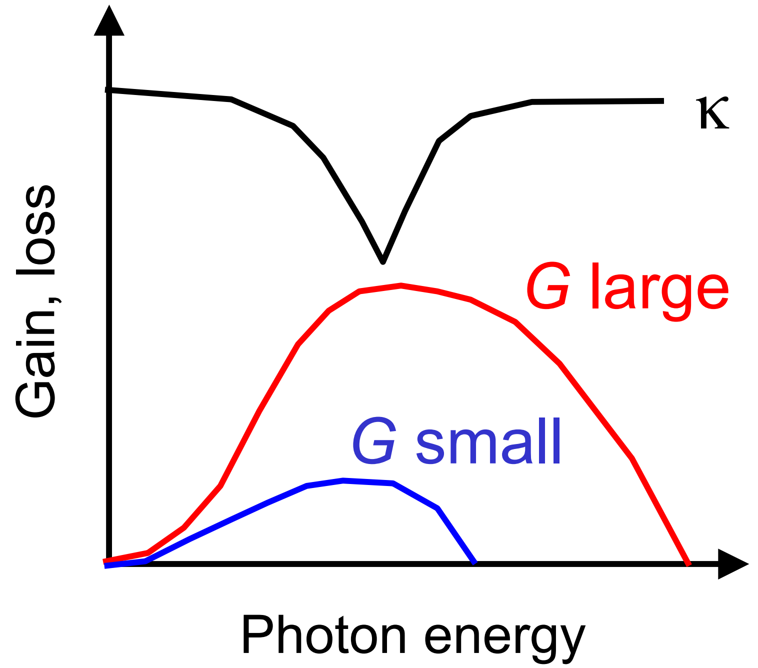
$$\frac{dn}{dt} = \frac{I}{e} - \gamma_e n - Gs = 0$$

$$\frac{ds}{dt} = (G\text{-}\kappa)s + \beta R_{sp} = 0$$

and

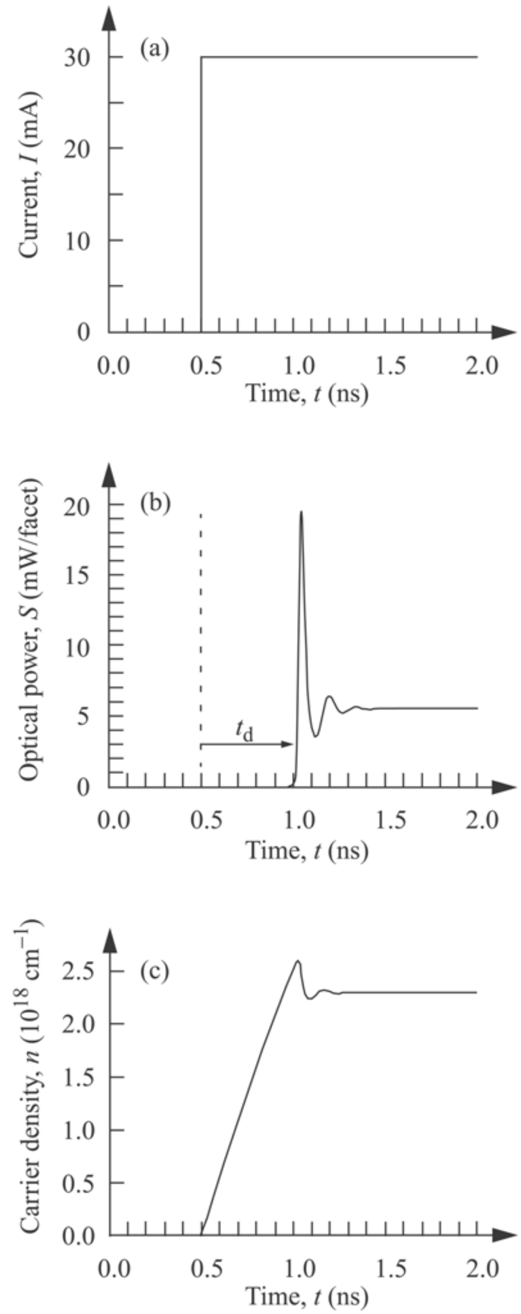
$$s = \frac{\beta R_{sp}}{(\kappa\text{-}G)}$$

As $G \rightarrow \kappa$ the number of photons in the system increases rapidly and *Gs* becomes large, so every extra electron injected into the system is converted into a photon, pinning the carrier density, *n*

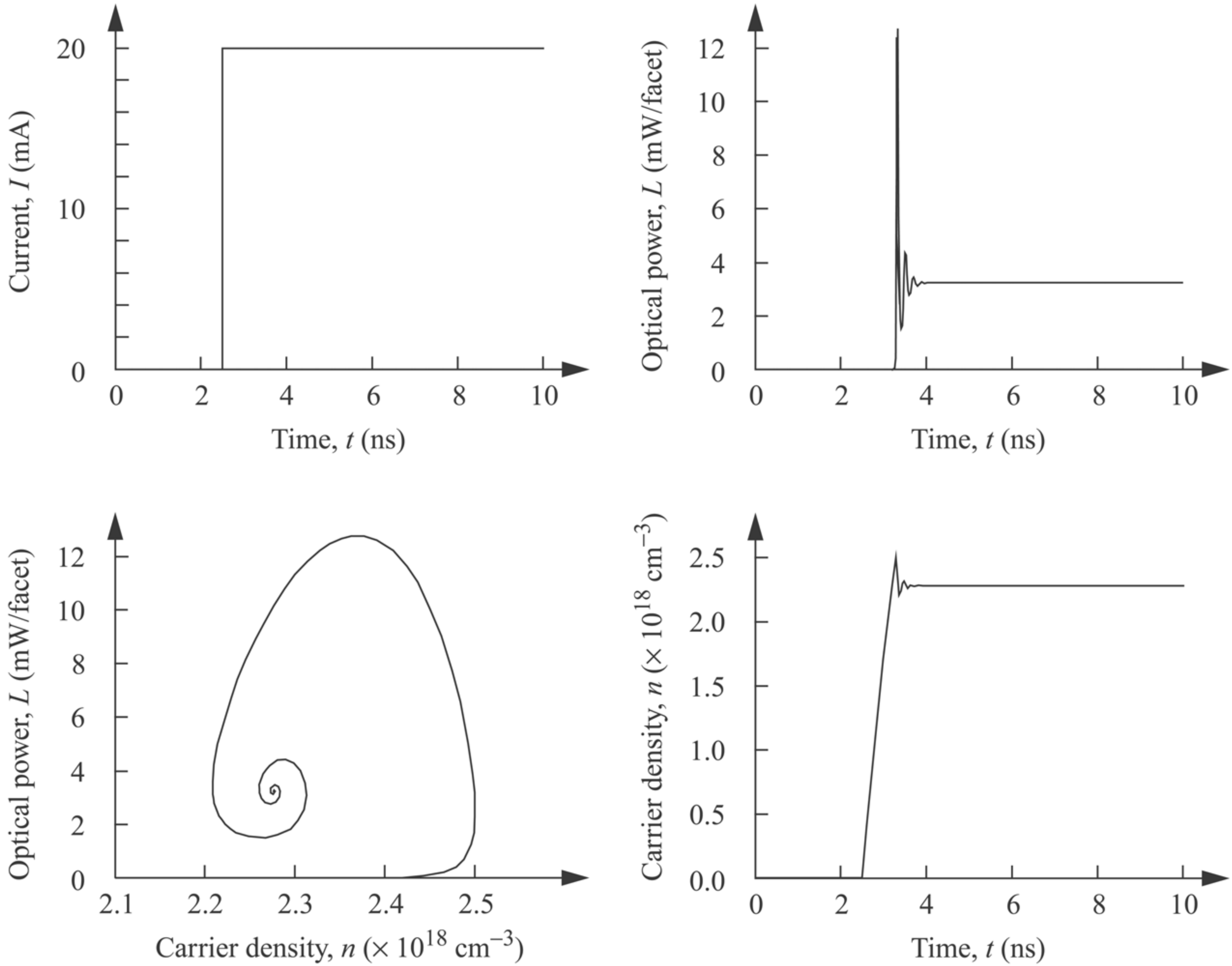


Continuum mean-field rate equation prediction

- Continuum mean-field rate equations predicted transient response to step change in current
  - Initial current I = 0 mA
  - Initial carrier number n = 0
- Light output
  - Turn-on delay, t_d
  - Relaxation oscillation
- Carrier density
  - n leads s
  - Overshoot
  - Relaxation oscillation

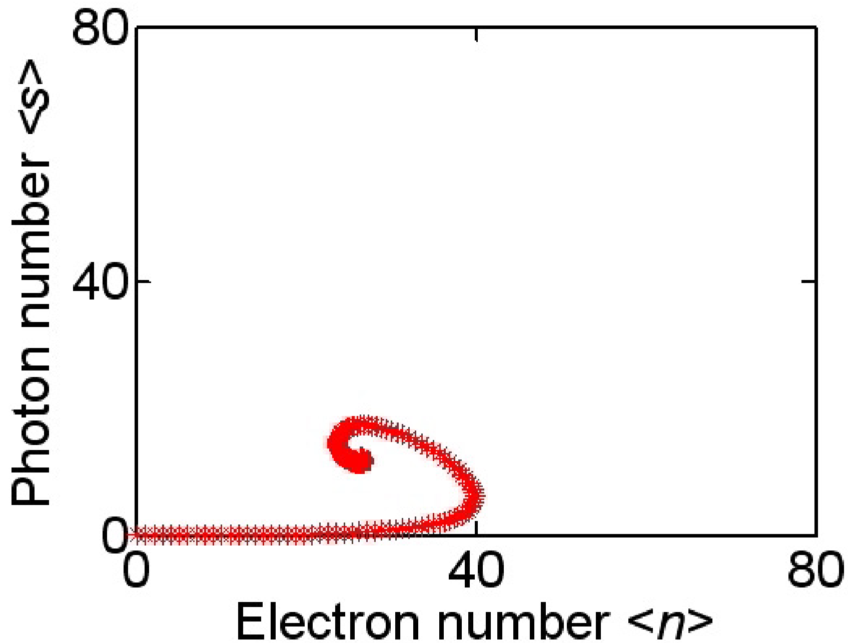


Continuum mean-field rate equation prediction



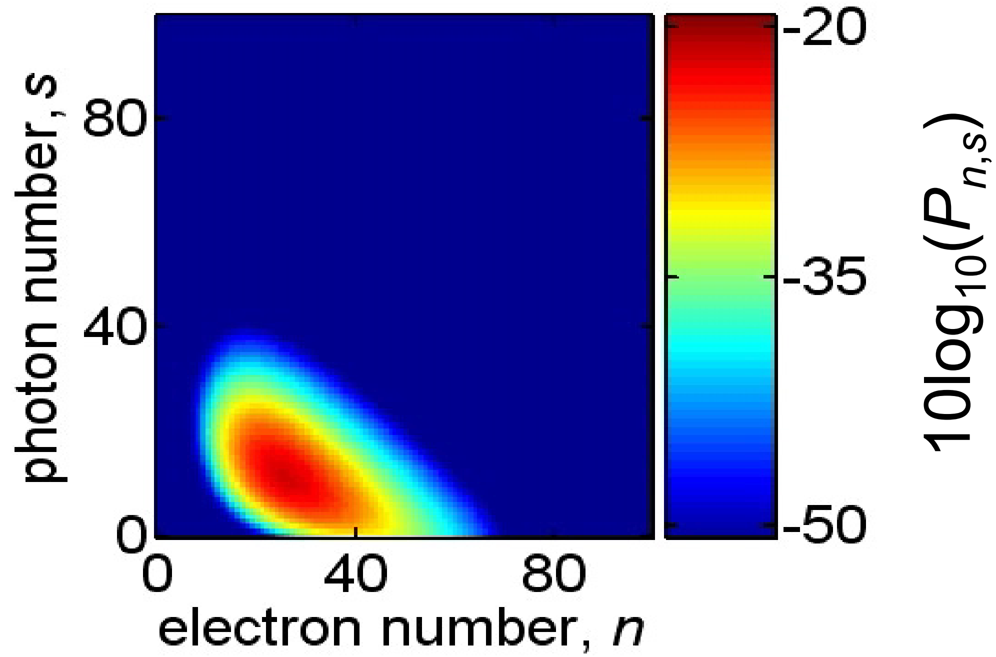
Mean-field versus probabilistic picture

Continuum mean-field rate equation (R.E.) prediction



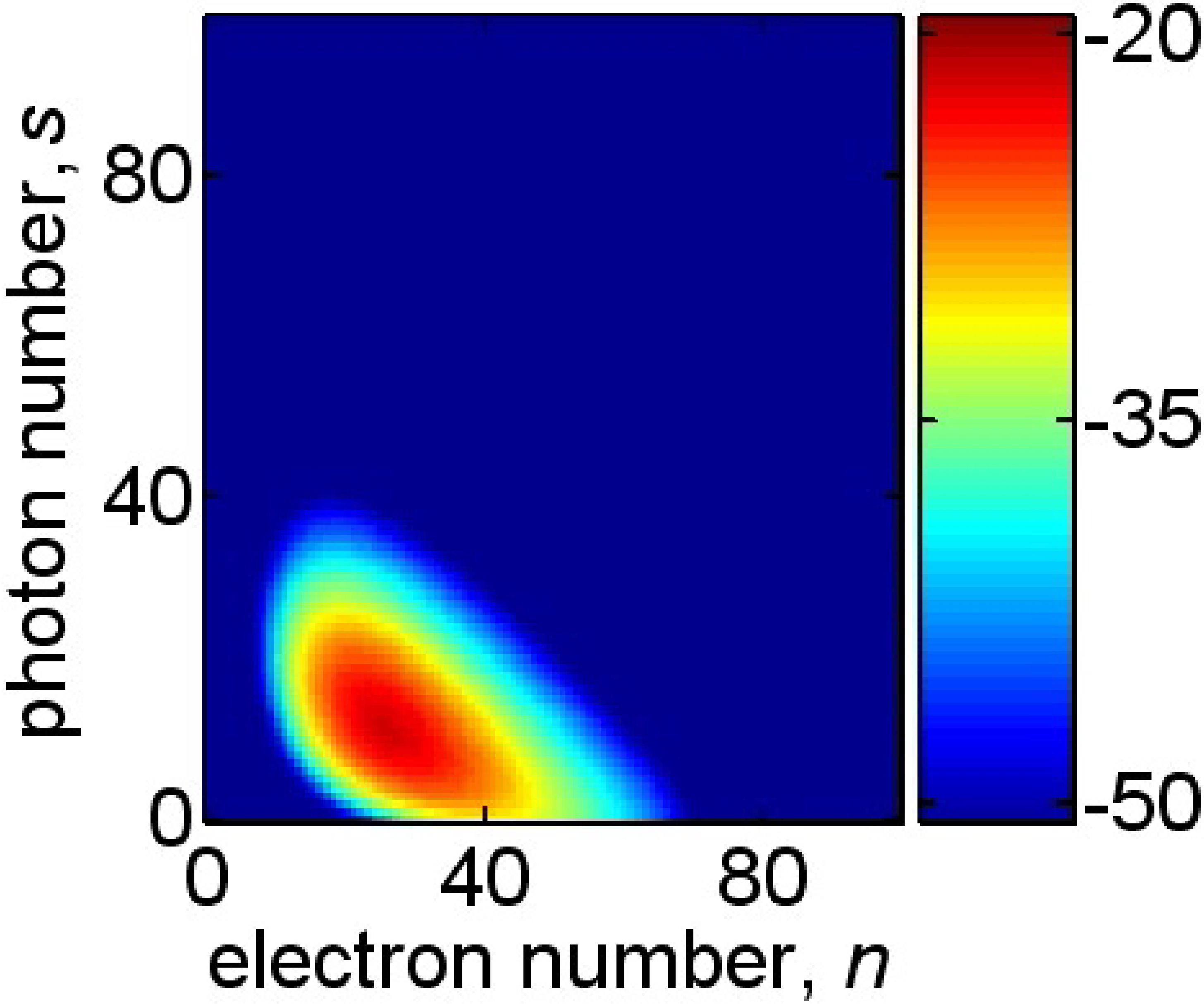
Approximate first moment (<n>, <s>) continuum mean – field calculation

Modeling discrete quantum system using continuum probability functions

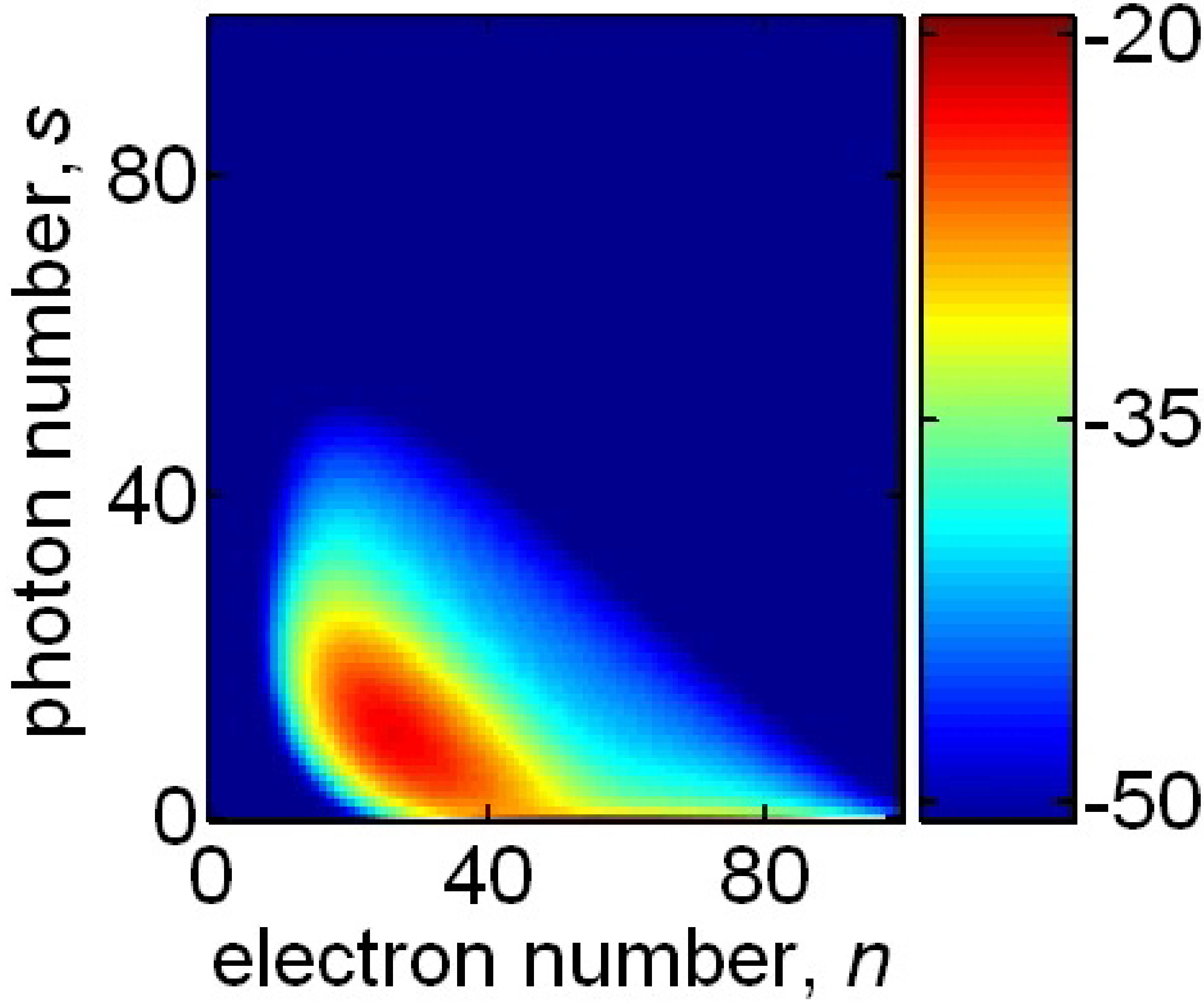


Probabilistic picture, $P_{n,s}$ for n electrons and s photons in the cavity

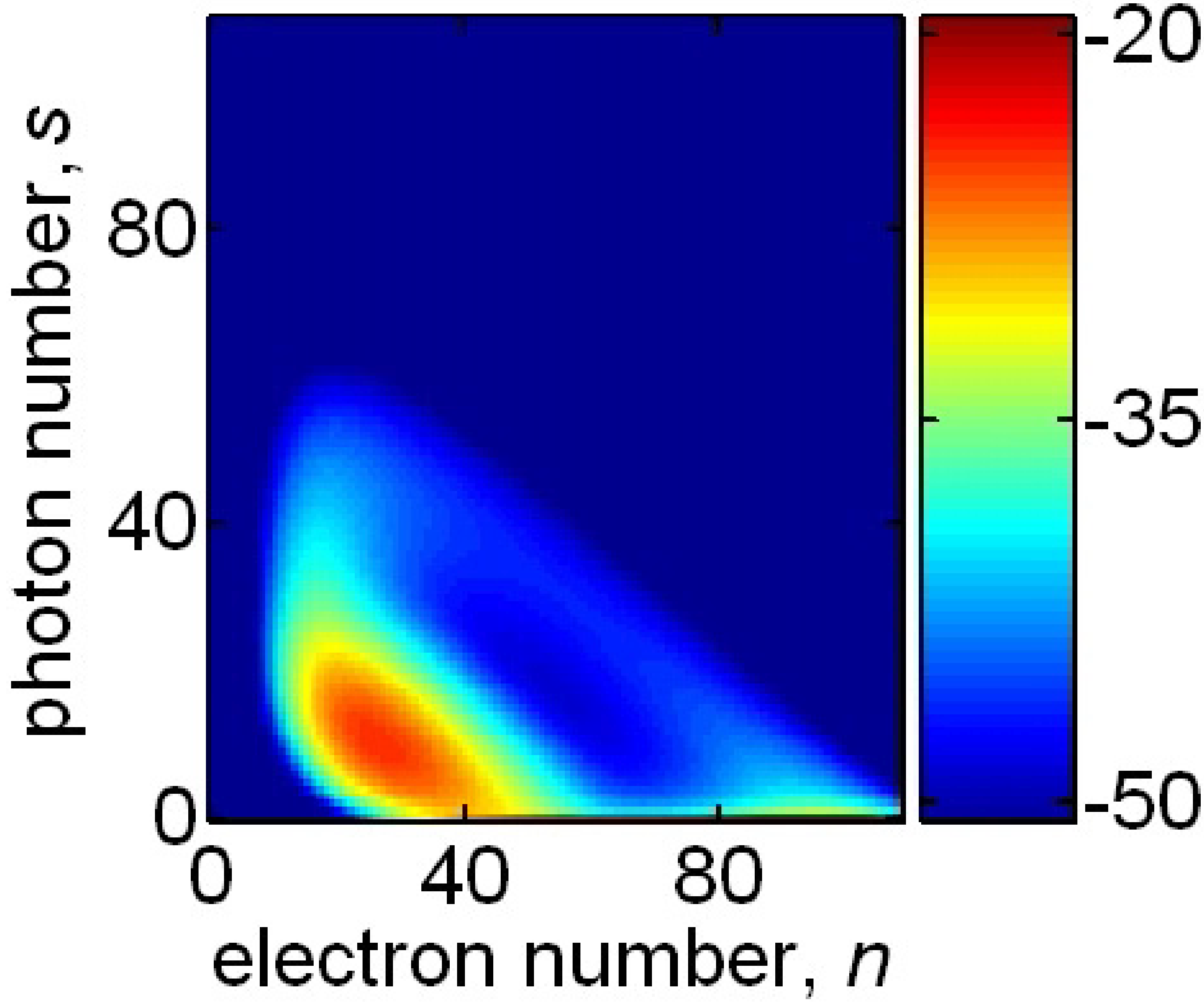
Time evolution of $10\log_{10}(P_{ns})$ for β=1



Time evolution of $10\log_{10}(P_{ns})$ for $\beta=0.1$

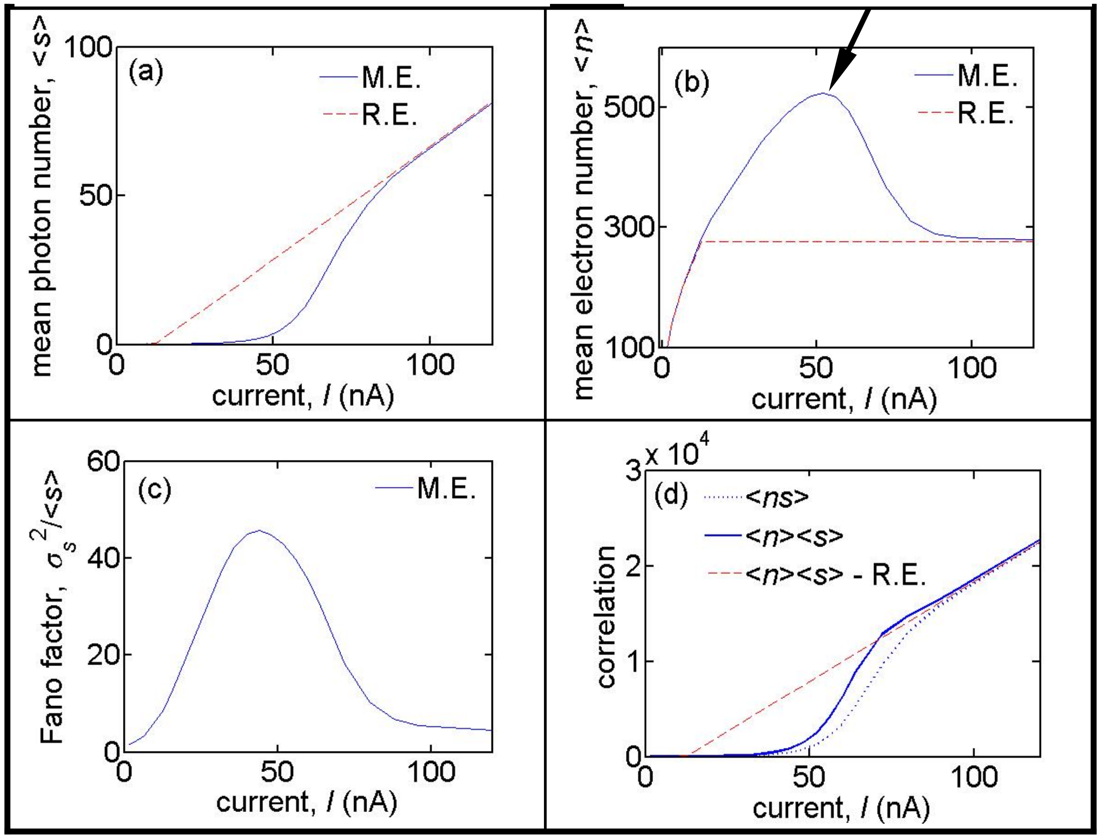


Time evolution of $10\log_{10}(P_{ns})$ for β=0.01



Master equation predictions for small cavity

Suppression of lasing due to fluctuations | **De-pinning of carriers**



<*ns*> does not factorize **(<*n*><*s*>)** in the **small cavity** limit leading to **suppression** of lasing and **de-pinning** of carriers.

Parameters : Volume = 0.1µm x 0.1µm x 10nm, $\Gamma = 0.25$, $a = 2.5 \times 10^{-18}$ cm^2 s^{-1}, $B = 10^{-10}$ cm^3 s^{-1}, $\beta = 10^{-4}$, $n_0 = 10^{18}$ cm^{-3}, $\alpha_i = 1$ cm^{-1}, $n_r = 4$, $r = 1 - 10^{-6}$.

Figure . (a) I = 9.6 nA. (b) I = 48 nA. (c) I = 72 nA. (d) I = 192 nA.

Summary so far for small lasers:

Fluctuations more important in small devices:

Master equations used to model discrete quantum system using continuum probability functions, P_{ns}

Predict suppression of lasing and de-pinning of carriers due to fluctuations (contrary to expectations of continuum mean-field rate equations)

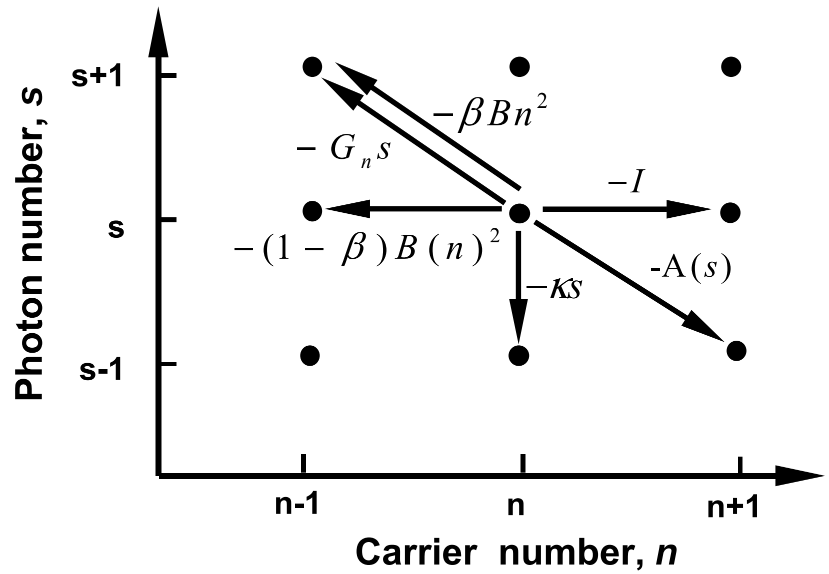
Need a different approach to understand the origin of these predictions (e.g. plot quantum trajectory)

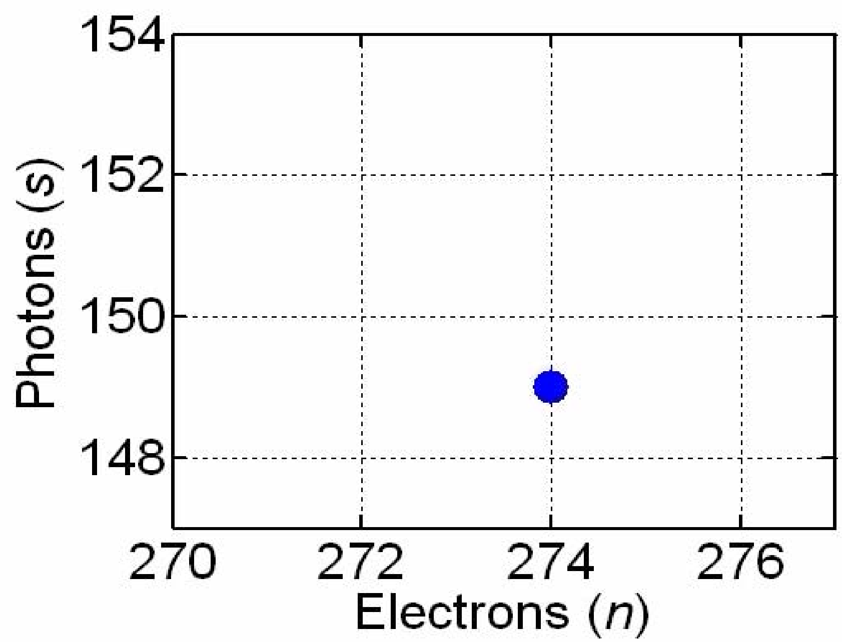
Laser simulation by Monte Carlo method

- At time t, system in state (n, s) has a choice to participate in six independent processes

- The time constants for different processes at time t, is estimated using the rate equation rates, e.g. photon cavity decay rate is ($\tau_{decay} = 1/\kappa s$)

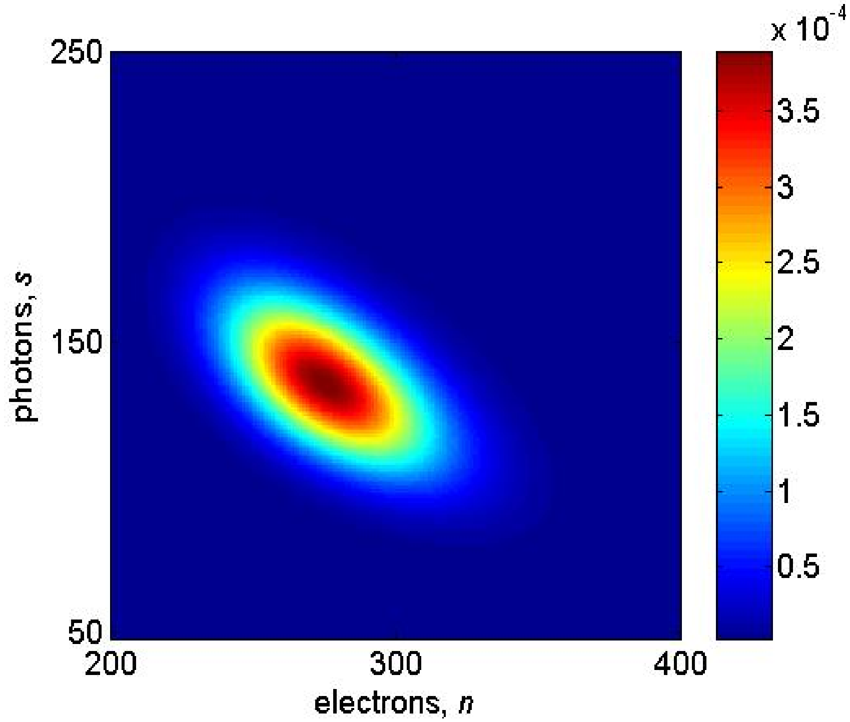
- System allowed to perform random walk on a 2D grid



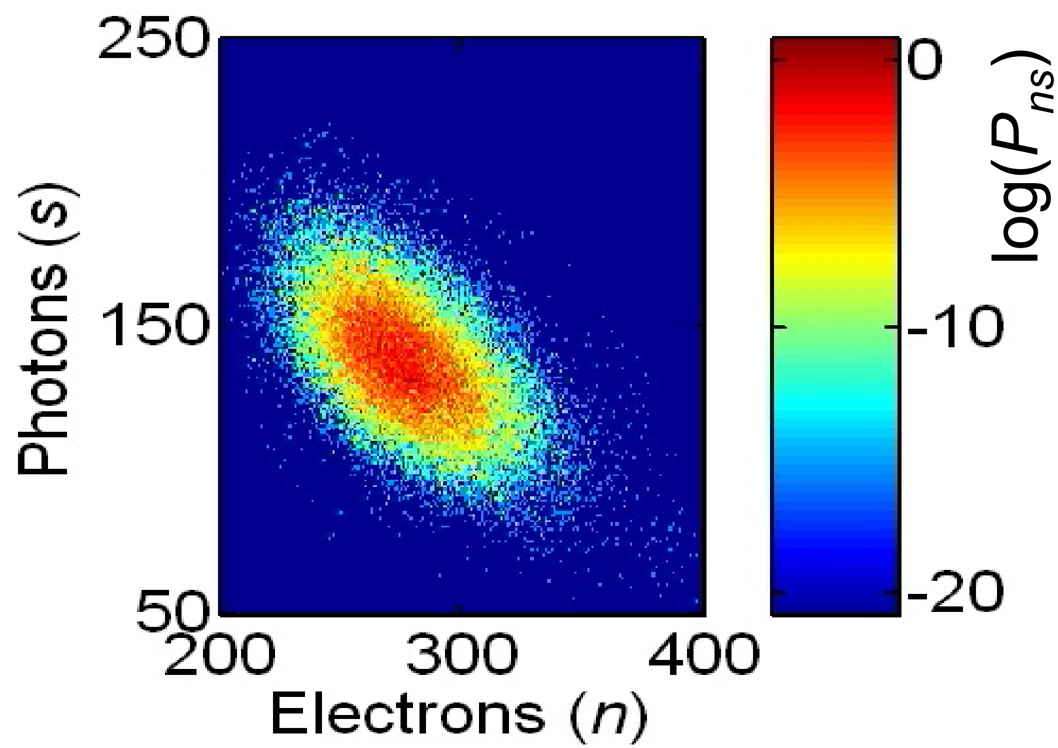


Laser simulation by Monte Carlo method

- Probabilities calculated from averages over multiple trajectories



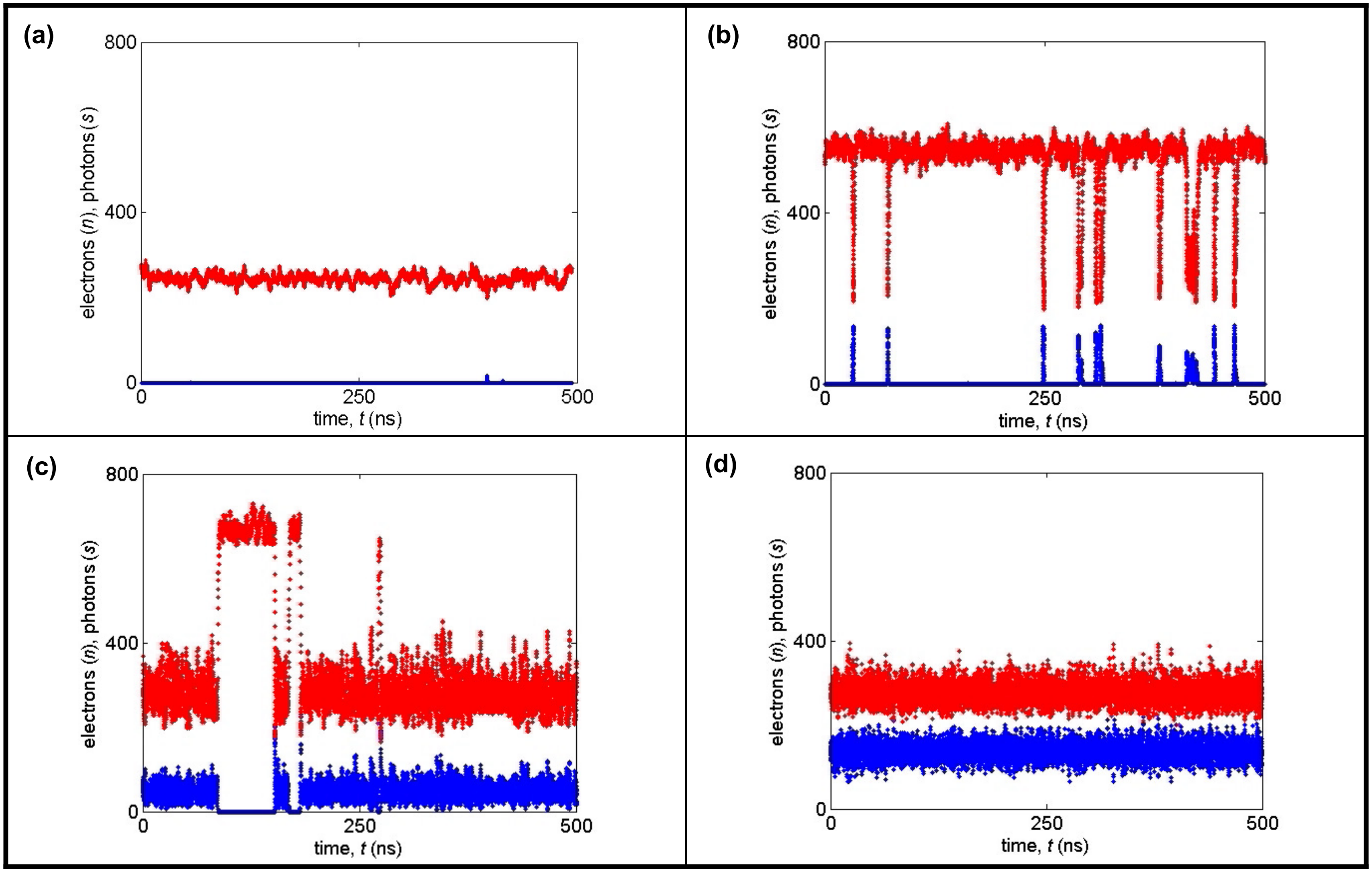
Steady state master equation solution to probability, $P_{n,s}$



Monte Carlo solution to probability, $P_{n,s}$

Parameters : Volume = 0.1μm x 0.1μm x 10nm, Γ = 0.25, a = 2.5 x 10^{-18} cm^2 s^{-1}, B = 10^{-10} cm^3 s^{-1}, β = 10^{-4}, n_0 = 10^{18} cm^{-3}, α_i = 1 cm^{-1}, n_r = 4, r = 1- 10^{-6}, I = 192 nA.

System trajectories by Monte Carlo method



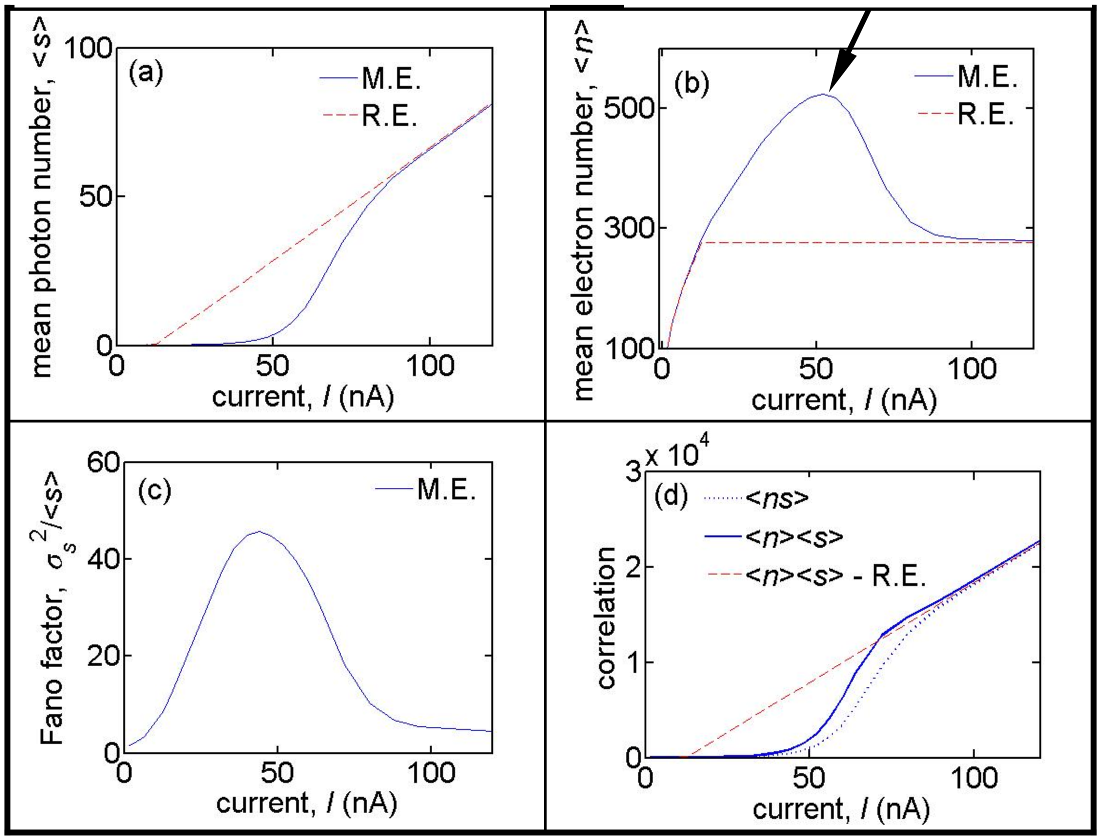
Parameters : Volume = 0.1μm x 0.1μm x 10nm, Γ = 0.25, a = 2.5 x 10^{-18} cm^2 s^{-1}, B = 10^{-10} cm^3 s^{-1}, β = 10^{-4}, n_0 = 10^{18} cm^{-3}, α_i = 1 cm^{-1}, n_r = 4, r = 1- 10^{-6}.

Figure . (a) I = 9.6 nA. (b) I = 48 nA. (c) I = 72 nA. (d) I = 192 nA. **Electrons (red), photons (blue).**

Master equation predictions for small cavity

Suppression of lasing due to fluctuations

De-pinning of carriers



<*ns*> does not factorize **(<*n*><*s*>)** in the **small cavity** limit leading to **suppression** of lasing and **de-pinning** of carriers.

Parameters : Volume = 0.1μm x 0.1μm x 10nm, Γ = 0.25, a = 2.5 x 10^{-18} cm^2 s^{-1}, B = 10^{-10} cm^3 s^{-1}, β = 10^{-4}, n_0 = 10^{18} cm^{-3}, α_i = 1 cm^{-1}, n_r = 4, r = 1- 10^{-6}.

Figure . (a) I = 9.6 nA. (b) I = 48 nA. (c) I = 72 nA. (d) I = 192 nA.

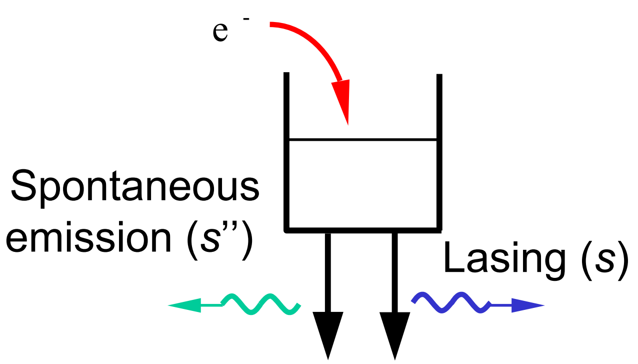
Master equation involving two emission modes

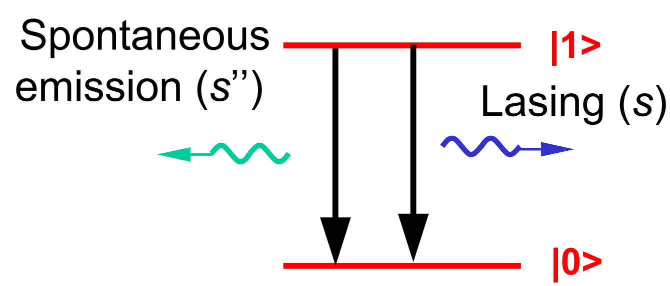
- **Must account for each electron – so if it is not creating lasing photons – where is it going?**

- Excess electrons create photons of another emission mode (p) which decays at the same rate as lasing photons and does not participate in any stimulated processes.

$$\langle P_{n,s}\rangle' = -\kappa(s P_{n,s,s''} - (s+1) P_{n,s+1,s''}) - (sG_n P_{n,s,s''} - (s-1)G_{n+1} P_{n,+1\,s-1,s''}) - (sAP_{n,s,s''} - (s+1)A P_{n,-1\,s+1,s''})$$
$$- \beta B(n^2 P_{n,s,s''} - (n+1)^2 P_{n+1,s-1,s''}) - (1-\beta)B(n^2 P_{n,s,s''} - (n+1)^2 P_{n+1,s,s''-1}) - I(P_{n,s} - P_{n-1,s})$$
$$-\kappa(s P_{n,s,s''} - (s+1) P_{n,s+1,s''+1})$$

where $P_{n,s,s''}$ is the probability of a state having n electrons, s lasing photons, s'' non-lasing photons.

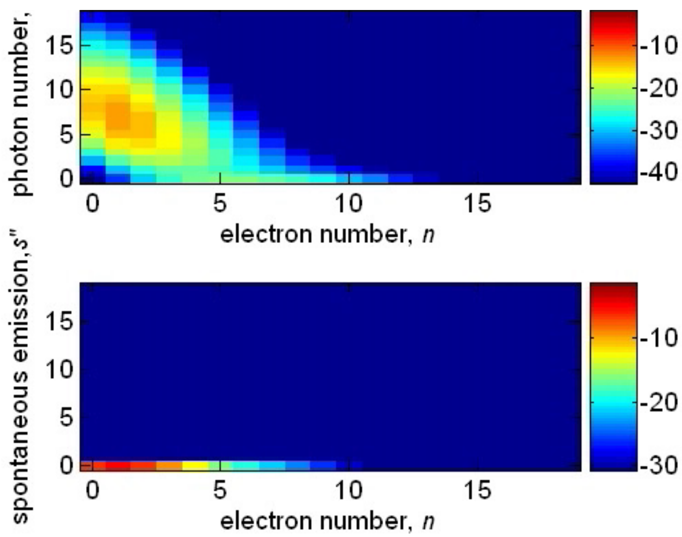




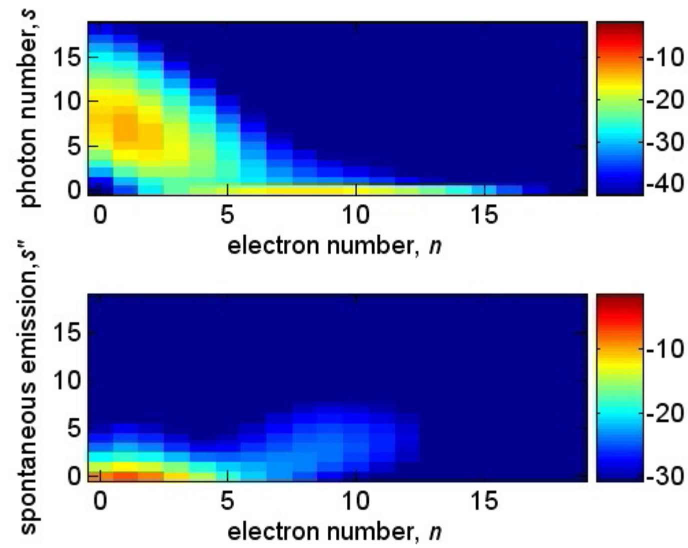
Probability distributions for different β

Parameters : (1μm*1nm*1nm) = 1e-18 cm^3, Γ = 0.25, a = 2.5e-018 cm^2/sec, B = 1e-10 cm^3/sec, n_g = 1e+18/ cm^3, α_l = 10 cm^{-1}, n_r = 4, r = 0.999, $\kappa=\kappa_{calc}$*1e-2, P = 10 electron/ns (= 1.6 nA) .

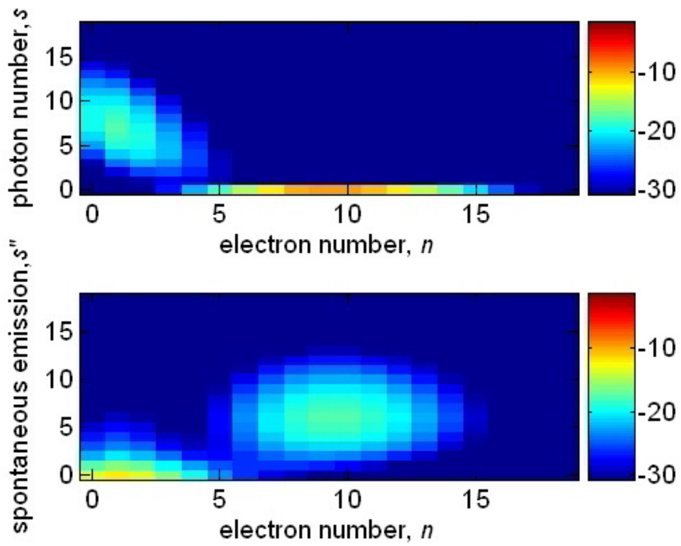
β = 1



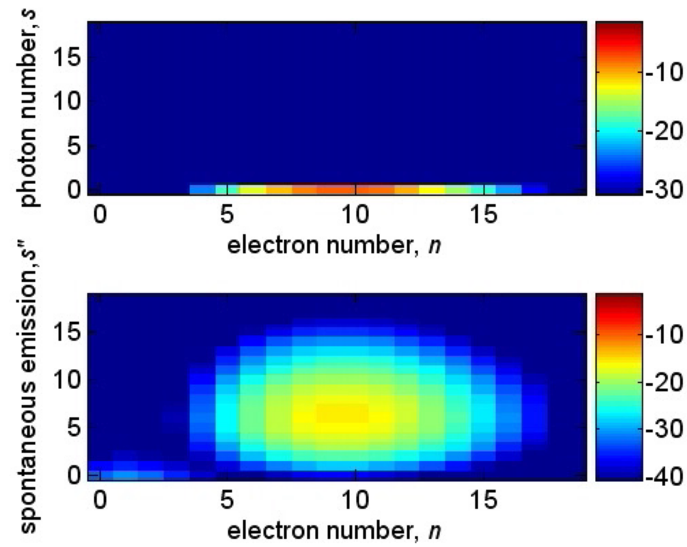
β = 0.1



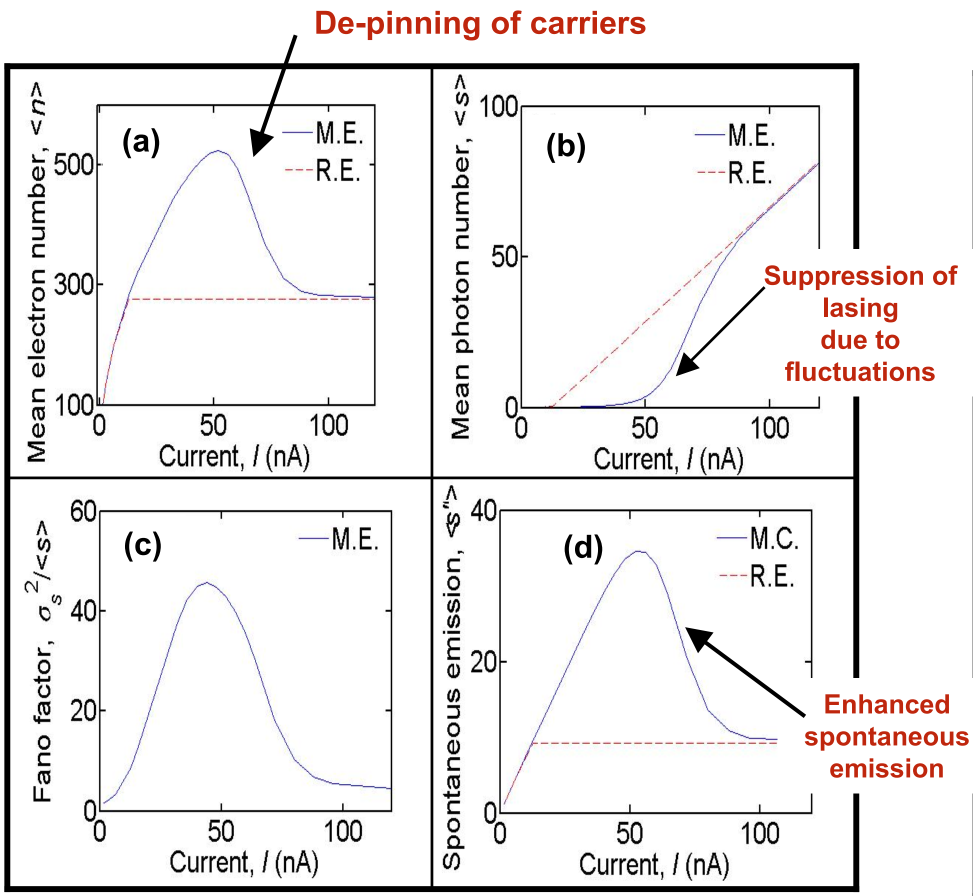
β = 0.01



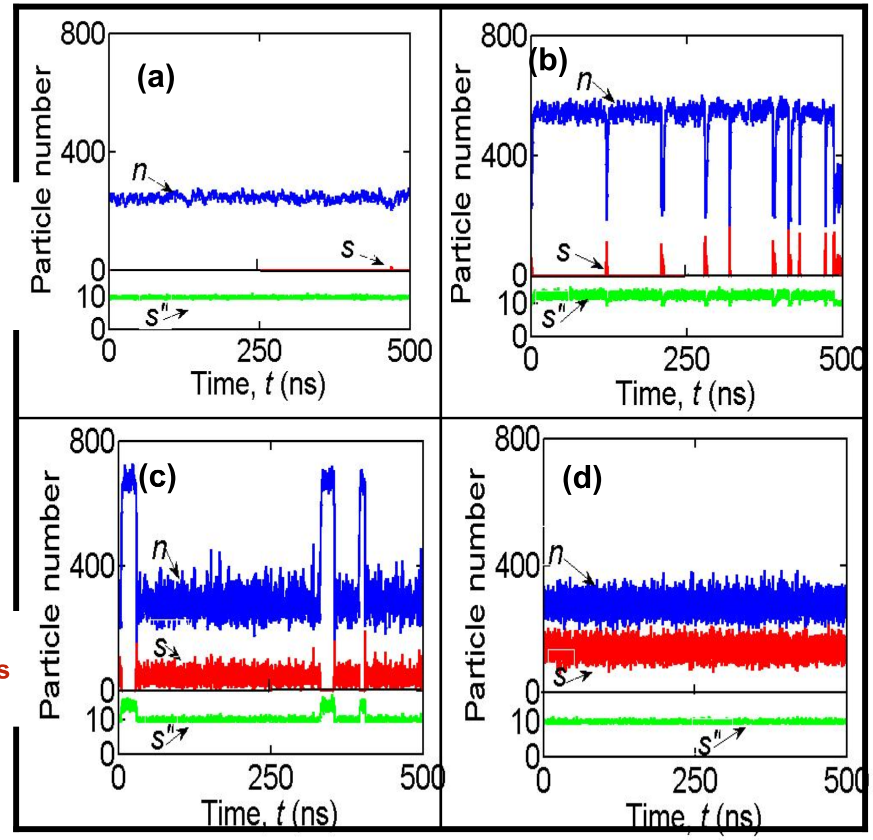
β = 0.0001



Comparison with random walk predictions



Master equation results

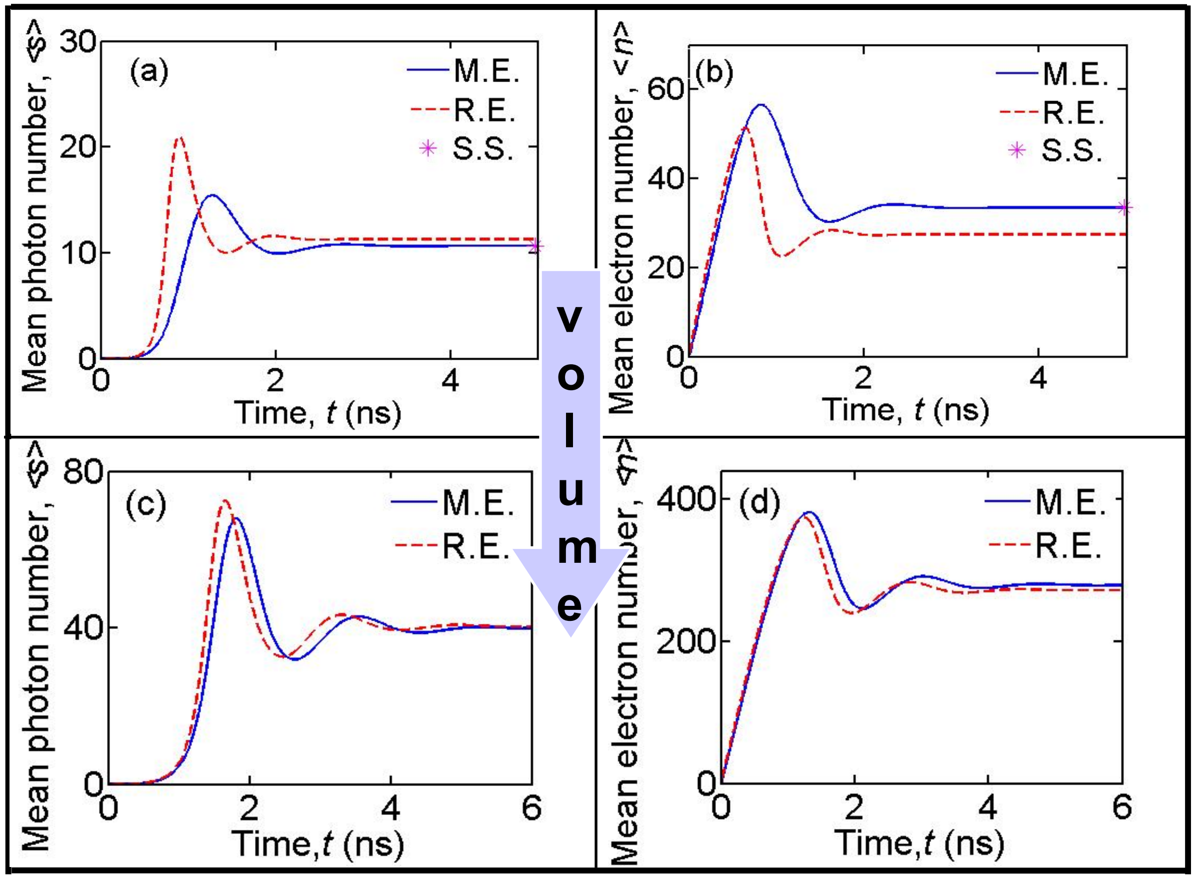


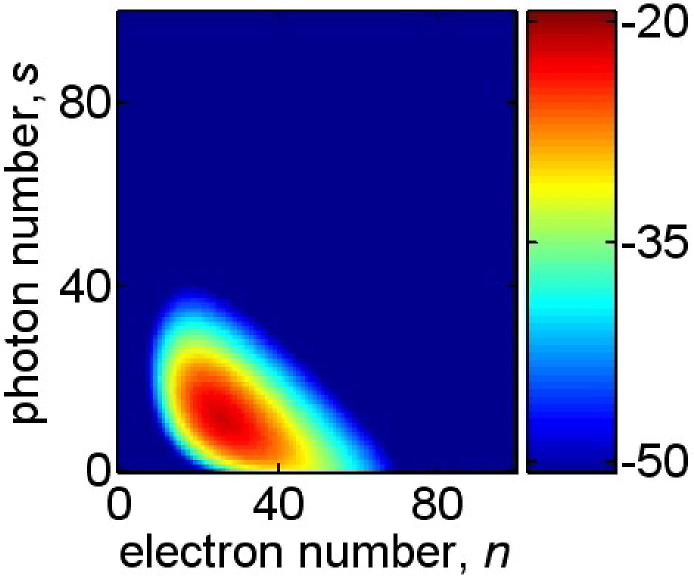
Monte Carlo results

Parameters : Volume = 0.1µm x 0.1µm x 10nm, $\Gamma = 0.25$, $a = 2.5 \times 10^{-18}$ cm² s⁻¹, $B = 10^{-10}$ cm³ s⁻¹, $\beta = 10^{-4}$, $n_0 = 10^{18}$ cm⁻³, $\alpha_i = 1$ cm⁻¹, $n_r = 4$, $r = 1- 10^{-6}$.

Figure . (a) I = 9.6 nA. (b) I = 48 nA. (c) I = 72 nA. (d) I = 192 nA.

Temporal characteristics with system size



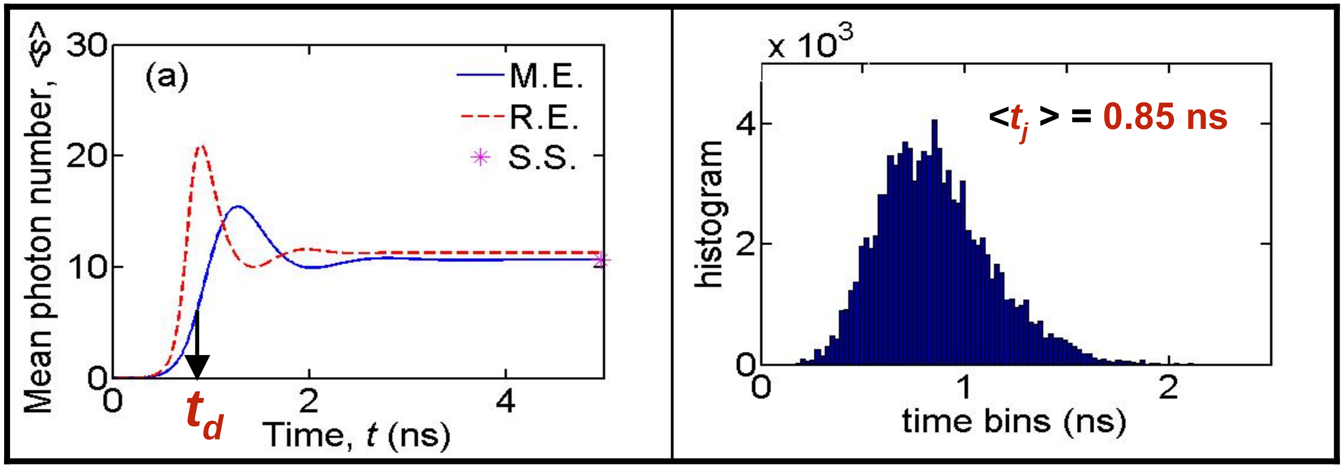


Parameters : L_c = 0.1μm, Γ = 0.25, a = 2.5e-018 cm^2/sec, B = 1e-10 cm^3/sec, β = 0.1, n_g = 1e+18/ cm^3, α_i = 10 cm^{-1}, n_r = 4, r = 0.999, $\kappa=\kappa_{calc}$*1e-2 .

Figure . Transient behavior of first moments of electrons and photons for P = 100 electron/ns (= 16 nA).
Transient Master equation results (blue) and rate equation results (red). (a) mean photon number vs time. (b) mean electron number vs time. (0.1μm*10nm*10nm = 1e-17 cm^3). (c) mean electron number vs time. (d) mean photon number vs time. (0.1μm* 0.1μm *10nm = 1e-16 cm^3).

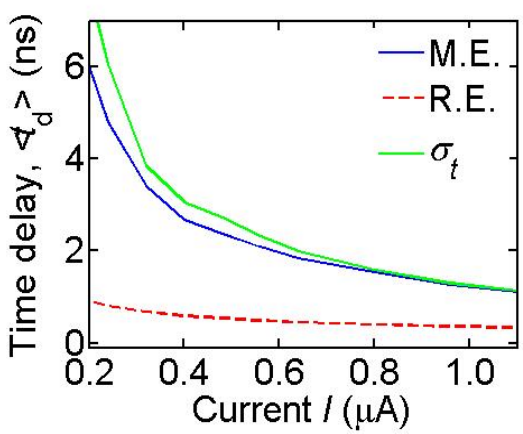
Temporal behavior – large signal analysis

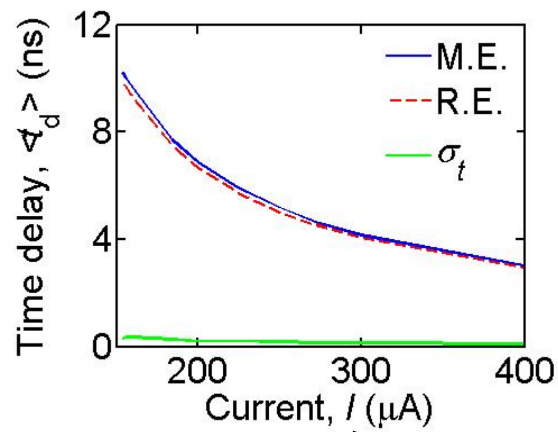
1. Response to a large step change in current (large signal)



Parameters :
Volume = 0.1μm x 10nm x 10nm,
Γ = 0.25, a = 2.5 x 10^{-18} cm^2 s^{-1},
B = 10^{-10} cm^3 s^{-1}, β = 10^{-1},
n_0 = 10^{18} cm^{-3}, α_i = 1 cm^{-1}, n_r = 4,
r = 1- 10^{-6}, I = 16 nA.

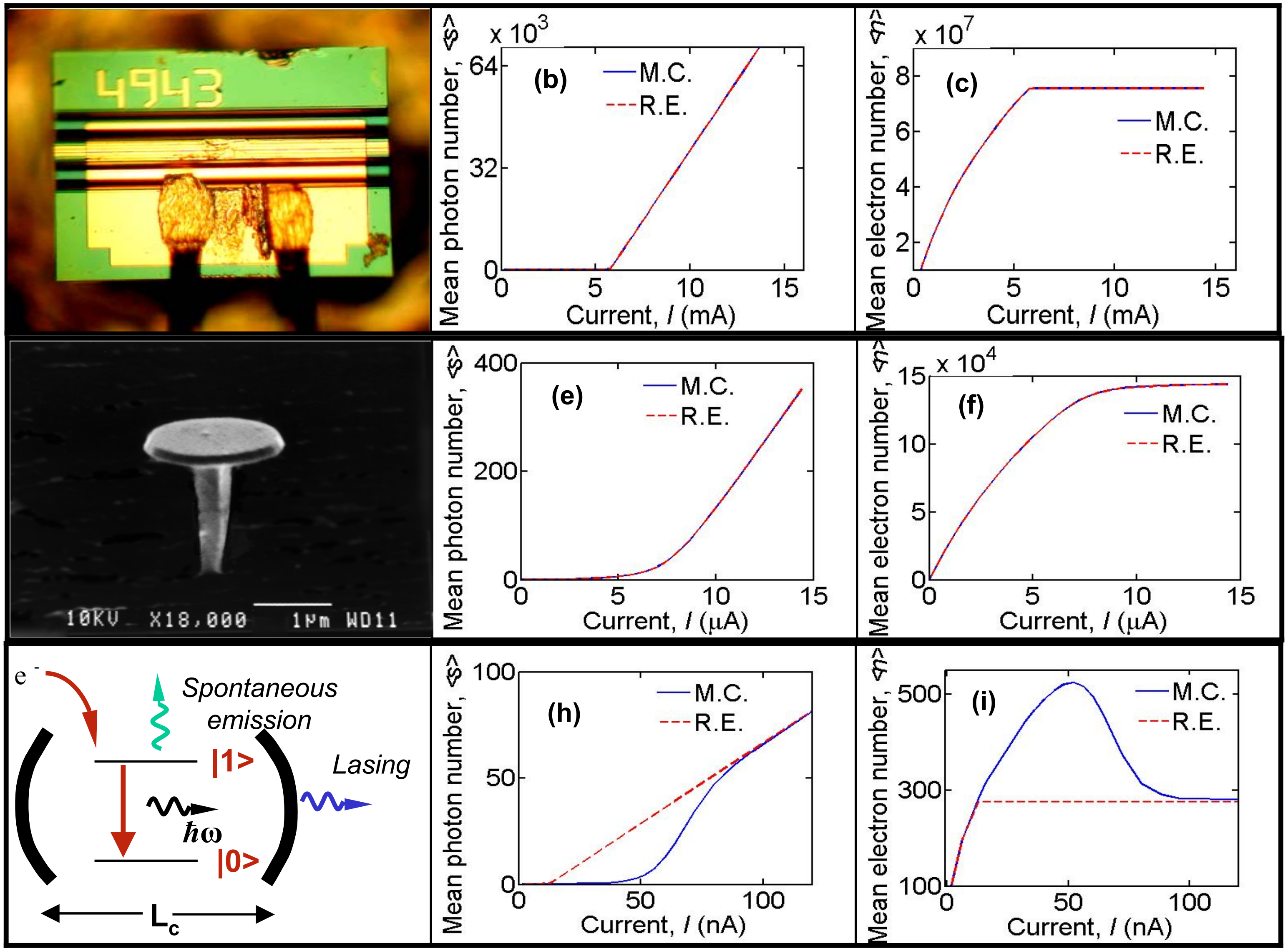
2. Time jitter (t_d) decreases with increasing pump





Volume

Summary of steady-state calculations



Volume = 34 μm^3
β = 5 x10^{-5}

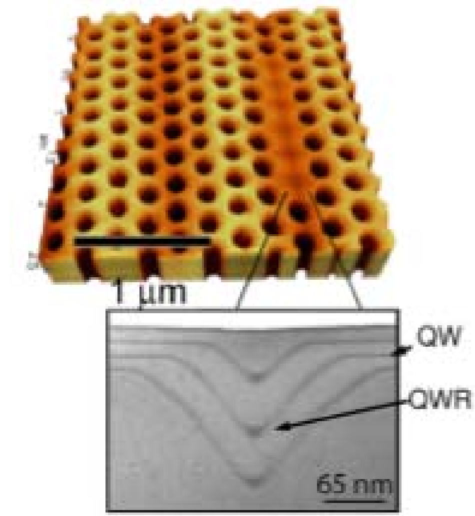
Volume = 0.12 μm^3
β = 10^{-1}

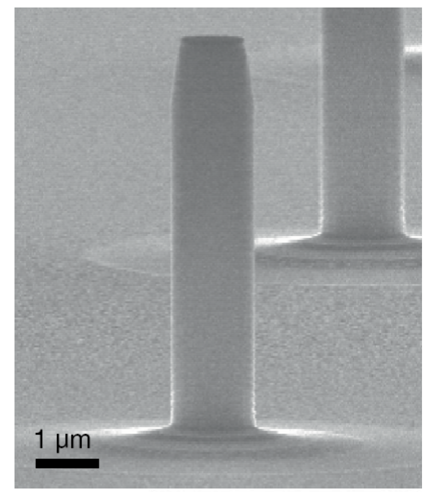
Volume = 10^{-4} μm^3
β = 10^{-4}

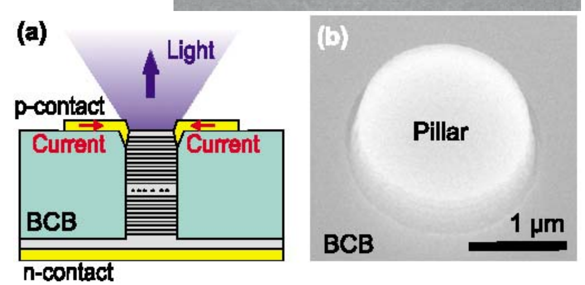
Figure . (a) Fabry – Perot laser. (d) Micro-disk laser. (g) Small laser (schematic diagram). (b), (e), (h) Mean photon number vs current. (c), (f), (i) Mean electron number vs current.

Making lasers with small active region volume

- **State-of-the-art quantum wire photonic crystal lasers and QD micropillar lasers can be made with very small active regions**
  - Active volume $V \sim 10^{-4}$ μm^3 can have $n \sim 100$ – 500
- **Examples:**
  - **EPFL**: Kirill Atlasov, Eli Kapon, et al., "Short (~1µm) Quantum-Wire Single-Mode Photonic-Crystal Microcavity Laser", CTuH4 CLEO/IQEC 2009
    - Estimate active volume $V \sim 3 \times 5\text{nm} \times 5\text{nm} \times 1000\text{nm} = 75 \times 10^{-18}\ \text{cm}^3 = 0.75 \times 10^{-4}\ \mu\text{m}^3$
    - Fluctuations important since $n \sim 100$ – 500
  - **U. Würzburg**: A. Forchel et al., "Single quantum dot controlled gain modulation in high-Q micropillar lasers", Phys. Status Solidi B **246**, No. 2, 277–282 (2009), Appl. Phys. Lett. **93**, 061104 (2008)
- **Quantum fluctuation effects should dominate device performance at low pump rates and $\beta < 10^{-2}$**

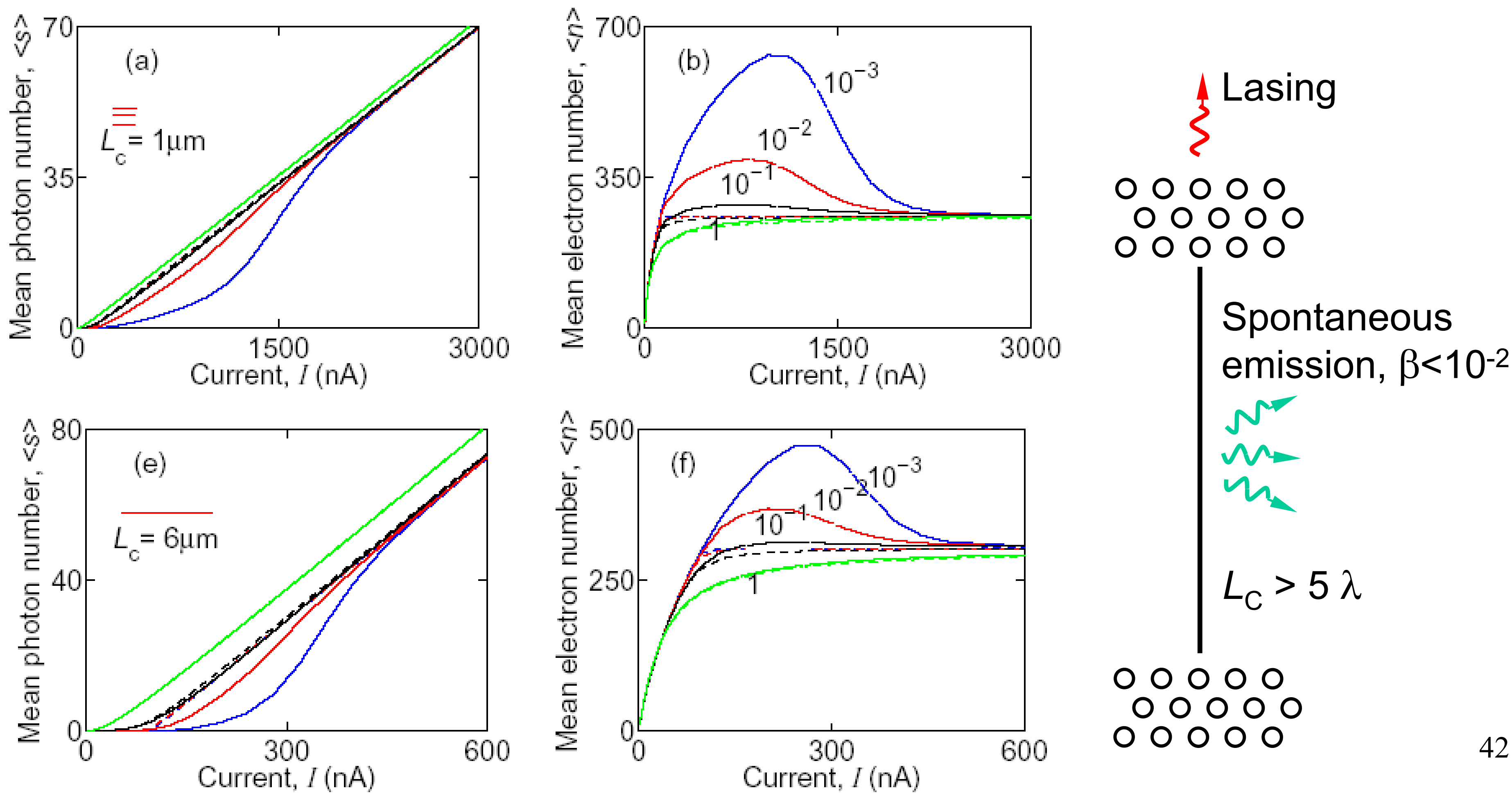






Making lasers with small active region volume

- **Calculations suggest that observation of both enhanced spontaneous emission and suppression of lasing due to quantum fluctuations requires $\beta < 10^{-2}$**



Conclusions

1. Fluctuations in *n* and *s* are important in determining the behavior of both large and small laser diodes
2. In large lasers fluctuations in *s* play an important role in determining the temperature dependence of laser threshold current
3. Quantum fluctuations in small lasers (and the fact that a ground state exists) can enhance spontaneous emission and suppress lasing near threshold (in contrast to predictions of Landau-Ginzburg in which fluctuations enhance emission below threshold). Dynamic switching between two characteristic system states dominates the fluctuations. Correlations between *n* discrete excited states and *s* discrete photons create a non-Poisson probability distribution and damp the average dynamic response of laser emission

Finite sized systems behave differently and, in particular, fluctuations are important !

Learn more: Phys. Rev. Lett. **102**, 053902 (2009)

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